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Robust Identification of 2-D Periodic Systems with Applications to Texture Synthesis and Classification

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***Abstract*— In this paper we address the problem of robust identification of separable in denominator 2-dimensional (2-D) discrete LTI systems that have a periodic impulse response. These systems arise in the context of many applications ranging from image processing to sensor arrays. The main result of the paper shows that a nominal plant that interpolates the experimental data as well as worst case bounds on the identification error can be obtained by performing a singular value decomposition on two Hankel matrices obtained from the experimental data. These results are illustrated with two practical examples arising in the context of image processing: texture synthesis and texture classification.**

I. Introduction

This paper addresses the problem of robust identification of separable in denominator 2-D discrete, quarter causal, shift invariant systems that have a periodic impulse response. This situation arises in the context of many practical applications from different areas, ranging from image processing to spatially distributed systems such as sensor arrays.

Robust identification of 1–D systems has been the object of considerably attention during the past 2 decades, and is by now a relatively mature field (see for instance [10], [15], [11], [2] and references therein). Regarding quarter-plane-casual, recursive and separable in denominator (CRSD) 2–D systems, several identification and model reduction algorithms have been proposed (see for instance [1], [7], [8], [9], [12], [13], [20], [21], [19] and references therein). However, none of these techniques can handle the case under consideration in this paper, where the plant is subject to the additional structure of having a periodic impulse response. In this paper, motivated by the above results and the work in [16] on identification of periodic 1–D systems, we propose a robust identification method for this class of systems. The proposed method is based upon a singular value decomposition (SVD) of two circulant Hankel matrices constructed from the experimental data, and is interpolatory (in the sense that it always yields a system inside the consistency set). Thus, well known results on information based complexity (see for instance Chapter 10 in [15]) can be brought to bear to compute worst case bounds on the identification error.

In the second portion of the paper we apply these tools to the problems of texture synthesis and texture classification. The main idea is to model images as the periodic impulse response of a quarter causal shift–invariant 2-D system and use the proposed method to identify the corresponding model. Partial or corrupted images can be completed by using the model to find the values of the missing pixels. Texture classification can be accomplished by recasting the problem into a 2-D model (in)validation form. By directly using a 2-D model, this approach avoids the difficulties encountered by methods based upon stacking the image row or column–wise and identifying a 1-D model ([16], [3]), where the identified model strongly depends on the chosen direction and can exhibit boundary artifacts due to artificially breaking patterns.

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II. Notation

In this paper we will consider quarter-causal, shift invariant, single–input single–output (SISO) 2-D systems, defined by a Roesser [14] state space model of the form:

$$\begin{aligned} x'(i,j) &= Ax(i,j) + Bu(i,j) \\ y(i,j) &= Cx(i,j) + Du(i,j) \end{aligned} \tag{1}$$

where

$$x'(i,j) = \begin{bmatrix} x^v(i+1,j) \\ x^h(i,j+1) \end{bmatrix}, x(i,j) = \begin{bmatrix} x^v(i,j) \\ x^h(i,j) \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

$x^v \in R^{n_1}, x^h \in R^{n_2}, u \in R$, and $y \in R$ denote the vertical and horizontal local states, a scalar input and a scalar output, respectively, and where A, B, C, and D are constant matrices of appropriate dimensions. Alternatively, we will represent this system by its convolution kernel $g(i,j)$. It is well known (see for instance [8]) that when $A_3 = 0$, the state space model above is separable in denominator (SDSS) (that is, the corresponding transfer matrix is of the form $G(z_1,z_2) = \frac{N(z_1,z_2)}{D_1(z_1)D_2(z_2)}$). In this case, the convolution kernel is related to the state–space matrices by the formulae:

$$g(i,j) = \begin{cases} C_1 A_1^{i-1} B_1, & i > 0, j = 0 \\ C_2 A_4^{j-1} B_2, & i = 0, j > 0 \\ C_1 A_1^{i-1} A_2 A_4^{j-1} B_2, & i > 0, j > 0 \end{cases} \tag{2}$$

In particular, we will consider a special class of systems of the form (1) having a *periodic* impulse response, that is, for all i, j the following property holds:

$$\begin{aligned} g(i+N,j) &= g(i,j) \\ g(i,j+M) &= g(i,j) \end{aligned} \quad \text{for some finite } N, M > 0 \tag{3}$$

In the sequel we will use the following two operators mapping $n \times m$ matrices to nm column vectors:

$$\begin{aligned} \mathbf{vec}(\mathbf{Y}) &= [y(1,1),\cdots,y(1,m),y(2,1),\cdots,y(n,m)]^T \\ \mathbf{mat}(\mathbf{y}) &= \begin{bmatrix} y(1,1) & \dots & y(1,m) \\ \vdots & \dots & \vdots \\ y(n,1) & \dots & y(n,m) \end{bmatrix} \end{aligned} \tag{4}$$

Given a data set $\{y(i,j),(1,1) \leq (i,j) \leq (n,m)\}$ we will associate to it the following *block circulant* Hankel matrix:

$$\mathbf{H}_y^{nm \cdot nm} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 & \dots & \mathbf{H}_n \\ \mathbf{H}_2 & \mathbf{H}_3 & \dots & \mathbf{H}_1 \\ \vdots & \vdots & \dots & \vdots \\ \mathbf{H}_n & \mathbf{H}_1 & \dots & \mathbf{H}_{n-1} \end{bmatrix} \tag{5}$$

where

$$\mathbf{H}_i = \begin{bmatrix} y(i,1) & y(i,2) & \dots & y(i,m) \\ y(i,2) & y(i,3) & \dots & y(i,1) \\ \vdots & \vdots & \dots & \vdots \\ y(i,m) & y(i,1) & \dots & y(i,m-1) \end{bmatrix}$$

In the sequel, we will denote by $\mathcal{H}$ the set of all block circulant Hankel matrices of the form (5). Finally, given a matrix $\mathbf{H}$, we will denote by $\|.\|_F$ its Frobenious norm, e.g., $\|\mathbf{H}\|_F^2 = \sum_{i,j} h(i,j)^2$.

III. Robust Identification of 2-D Periodic Systems

A. Statement of the Problem

Consider the problem of identifying a periodic 2-D plant G from measurements of its output $y(i,j)$, $(0,0) \leq (i,j) \leq (n-1,m-1)$, corrupted by additive bounded noise v in a given set $\mathcal{N}$, to a known input u applied in some past interval $[-T_v,-1] \times [-T_h,-1]$.

$$\begin{gathered} y(i,j) = \sum_{k=-T_v}^{-1} \sum_{l=-T_h}^{-1} g(i-k,j-l)u(k,l) + v(i,j), \\ (0,0) \leq (i,j) \leq (n-1,m-1); \quad v \in \mathcal{N}. \end{gathered} \tag{6}$$

Further, the plant is known to have a separable in denominator state–space (SDSS) realization where the dimensions of the vertical and horizontal states are bounded by $\dim\{x^v\} \leq r$, $\dim\{x^h\} \leq s$, respectively, and to satisfy the structural constraints, $g(i,j) = g(i+N,j)$ and $g(i,j) = g(i,j+M)$, where N and M are given.

Assume now that the input signal is applied for an integer number of periods, e.g. $T_v = \alpha N$ and $T_h = \beta M$. From the periodicity assumption it follows that:

$$\begin{aligned} y(i,j) &= \sum_{k=-T_v}^{-1} \sum_{l=-T_h}^{-1} g(i-k,j-l)u(k,l) + v(i,j) \\ &= \sum_{k=-N}^{-1} \sum_{l=-M}^{-1} g(i-k,j-l)\tilde{u}(k,l) + v(i,j); \\ \tilde{u}(k,l) &\doteq \sum_{i_1=0}^{\alpha-1} \sum_{j_1=0}^{\beta-1} u(k-i_1N, l-j_1M); \end{aligned} \tag{7}$$

Thus, without loss of generality, it will be assumed in the sequel that $T_v = N$ and $T_h = M$. In addition, in the sequel we will make the following assumptions:

A1.- $n = N$ and $m = M$, that is, a full period of output data is available.

A2.- The input $u(.,.)$ is chosen so that $\mathbf{H}_u^T\mathbf{H}_u = \mathbf{I}$ and the measurement noise $v(.,.)$ satisfies the inequality $\|\mathbf{H}_v\|_F \leq \sqrt{n \cdot m} \cdot \epsilon$.

Remark 1: Since the matrices $\mathbf{H}$ are block circulant, it can be shown that their singular values are given by the magnitude of the Fourier Transform of their first row. Thus, assumption A2 above is a deterministic equivalent of selecting white noise as the experimental input, and having stochastic measurement noise with energy bounded by ϵ^2.

With these assumptions the identification problem of interest here can be precisely stated as follows.

Problem 1: Given:

(i) *a priori* set descriptions of the measurement noise $\mathcal{N}$ and candidate models $\mathcal{S}$:

$$\begin{aligned} \mathcal{S} &\doteq \{\text{SDSS impulse response periodic 2D systems}: \\ &\quad \dim\{x^v\} \leq r, \dim\{x^h\} \leq s, \\ &\quad g(i+N,j+M) = g(i,j)\} \\ \mathcal{N} &\doteq \{v : \|\mathbf{H}_v\|_F \leq \sqrt{n \cdot m} \cdot \epsilon\} \end{aligned}$$

(ii) a finite set of samples of the input $\{u(i,j)\}$, $(-N,-M) \leq (i,j) \leq (-1,-1)$ and the corresponding output $\{y(i,j)\}$, $(0,0) \leq (i,j) \leq (N-1,M-1)$, $(i,j) \neq (0,0)$.

Then:

a.- Determine whether the consistency set $\mathcal{T}(y)$ is nonempty, where

$$\begin{aligned} \mathcal{T}(y) \doteq \{g \in \mathcal{S} : \{y - (g*u)\}_{(i,j)} \in \mathcal{N}, \\ (0,0) \leq (i,j) \leq (N-1,M-1)\} \end{aligned}$$

b.- If $\mathcal{T}(y) \neq \emptyset$, find a model $g_{id} \in \mathcal{T}(y)$ and a bound on the worst-case identification error.

IV. An Identification Algorithm for 2-D Periodic Systems

A. Problem Solution

In this section, we present a solution to Problem 1, based upon the singular value decomposition of two circulant Hankel matrices constructed from the experimental data. To this effect, consider the following algorithm:

Algorithm 1:

1.- Given the experimental data $\{u(i,j), y(i,j)\}$, form the vector $\mathbf{u} \doteq \mathbf{vec}\,(u(i,j))$ and the (circulant) Hankel matrix $\mathbf{H}_y$. Let $\mathbf{Y}_o \doteq \mathbf{mat}(\mathbf{H}_y\mathbf{u})$ and perform the singular value decomposition

$$\mathbf{Y_0} = \begin{bmatrix} U & U_\perp \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V^T \\ V_\perp^T \end{bmatrix},$$
$$\Sigma = \text{diag}(\sigma_1, \cdots, \sigma_p), \sigma_i \geq \sigma_j > 0,\ i \geq j$$

2.- Define the matrices

$$F \doteq U\Sigma^{1/2}W \in R^{N \times p};\ G \doteq W^T\Sigma^{1/2}V^T \in R^{p \times M}$$

where $W \in R^{p \times p}$ is an arbitrary orthogonal matrix. Let $f_i \in R^{1 \times p}$ and $g_j \in R^{p \times 1}$ denote the i^{th} row and j^{th} column of F and G respectively.

3.- Form the following *block circulant* Hankel Matrix:

$$H_f \doteq \begin{bmatrix} f_1 & f_2 & \cdots & f_N \\ f_2 & f_3 & \cdots & f_1 \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_1 & \cdots & f_{N-1} \end{bmatrix} \tag{8}$$

Perform a singular value decomposition:

$$H_f = \begin{bmatrix} U_f & U_f^\perp \end{bmatrix} \begin{bmatrix} S_f & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_f^T \\ (V_f^\perp)^T \end{bmatrix},$$
$$S_f = \text{diag}(\sigma_1^f, \ldots, \sigma_{n_f}^f),\ \sigma_i^f \geq \sigma_j^f > 0,\ i \geq j \tag{9}$$

and define the matrices:

$$\begin{aligned} A_f &= S_{r,f}^{-\frac{1}{2}} U_{r,f}^T P_N U_{r,f} S_{r,f}^{\frac{1}{2}},\ B_f = S_{r,f}^{\frac{1}{2}} V_{r,f}^{(1)} \\ C_f &= U_{r,f}^{(1)} S_{r,f}^{\frac{1}{2}},\ D_f = 0 \end{aligned} \tag{10}$$

where

$$P_N = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix} \in R^{N \times N} \tag{11}$$

and $S_{r,f} = \text{diag}(\sigma_1^f, \ldots, \sigma_r^f)$, $U_{r,f}$ ($V_{r,f}$) denotes the sub-matrix formed by the first r columns (rows) of U_f (V_f), $U_{r,f}^{(1)}$ and $V_{r,f}^{(1)}$ denote the first $1 \times r$ block of $U_{r,f}$ and $r \times p$ block of $V_{r,f}^T$, respectively.

4- Applying a similar procedure to the matrix G, obtain a triple (A_g, B_g, C_g):

$$\begin{aligned} S_g &= \text{diag}(\sigma_1^g, \ldots, \sigma_{n_g}^g),\ \sigma_i^g \geq \sigma_j^g > 0,\ i \geq j \\ A_g &= S_{s,g}^{-\frac{1}{2}} U_{s,g}^T P_{Mp} U_{s,g} S_{s,g}^{\frac{1}{2}},\ B_g = S_{s,g}^{\frac{1}{2}} V_{s,g}^{(1)} \\ C_g &= U_{s,g}^{(1)} S_{s,g}^{\frac{1}{2}} \end{aligned} \tag{12}$$

Theorem 1: If the following condition holds:

$$\begin{aligned} & e_1 + e_2 + e_3 \\ & \doteq \bar{\sigma}(\mathbf{Y}_0) \left(\frac{\sum_{i=r+1}^{n_f} (\sigma_i^f)^2}{N} \right)^{1/2} \\ & + \bar{\sigma}(\mathbf{Y}_0) \left(\frac{\sum_{i=s+1}^{n_g} (\sigma_i^g)^2}{M} \right)^{1/2} \\ & + \sqrt{p} \left(\frac{\sum_{i=r+1}^{n_f} (\sigma_i^f)^2}{N} \right)^{1/2} \left(\frac{\sum_{i=s+1}^{n_g} (\sigma_i^g)^2}{M} \right)^{1/2} \\ & \leq \epsilon \end{aligned} \tag{13}$$

then:

(i) The consistency set $\mathcal{T}(y) \neq \emptyset$

(ii) The SDSS 2D system with state space realization

$$\begin{aligned} A_{id} &= \begin{bmatrix} A_f & B_f C_g \\ 0 & A_g \end{bmatrix};\ B_{id} = \begin{bmatrix} 0 \\ B_g \end{bmatrix}, \\ C_{id} &= \begin{bmatrix} C_f & 0 \end{bmatrix};\ D_{id} = 0 \end{aligned} \tag{14}$$

interpolates the experimental data within the measurement noise level and satisfies the required periodicity constraint.

Proof: Given in the Appendix ■

B. Analysis of the Identification Error

Since the proposed algorithm is interpolatory and the *a priori* sets $\mathcal{S}$ and $\mathcal{N}$ are convex, symmetric with respect to $(0,0)$ in the 2-D domain, from Lemma 10.4 in [15] the worst case identification error satisfies:

$$\|e\|_F \leq 2 \sup_{g \in \mathcal{T}(0)} \|g\|_F$$

where

$$\mathcal{T}(0) = \{g \in \mathcal{S} : \mathbf{H}_g\mathbf{H}_u + \mathbf{H}_v = \mathbf{0},\ \text{for some}\ v \in \mathcal{N}\}$$

Since $\mathbf{H}_u$ is unitary, it follows that $\|e\|_F \leq 2\epsilon$.

V. Applications: Texture synthesis and classification

Texture modelling has been a long standing problem in computer vision. Statistical approaches proceed by modelling texture as a stochastic process and attempting to capture the relevant properties [4], [6]. A related approach was proposed in [17], [16], where textured images were modelled as the output of a 1-D linear shift-invariant system driven by white noise. While successful in many situations, this approach required stacking the image either row or column–wise, leading to different models depending on the selection, and potentially introducing boundary artifacts. To avoid this difficulty, in this paper we will model images directly as the output of a shift–invariant, 2D system $G(.,.)$, driven by white noise, reducing the problem to that of identifying the relevant system model from the given images. Once these models are obtained, synthesis follows in a straightforward fashion, by simply driving the model with a suitable input. Similarly, texture classification can be accomplished by interrogating a collection of models to determine which one is the closest (in a sense that we will precisely define later) to the given sample.

A. Texture modelling and synthesis

A potential difficulty with the approach outlined above is that the system associated with a given texture is not necessarily causal since the intensity value at a pixel is likely to depend on the values of all pixels in its neighborhood, not just on those preceding it in some ordering of the image.

Motivated by the work in [17], [16], we propose to circumvent this difficulty by considering a given $n \times m$ image as one period of an infinite 2D signal with period (n, m). Thus, at any given location (i, j) in the image, the intensity values $\mathcal{I}(r, s)$ at other pixels are available also at position $(r - qn, s - qm)$, and the integer q can always be chosen so that $r - qn < i, s - qm < j$. From this observation, it follows that the unknown operator $G(.,.)$ admits a state space representation of form (1), subject to an additional constraint of the form (3). Finally, without loss of generality, we can assume that the specific image being considered corresponds to the case $u = \delta(0, 0)$, by absorbing the dynamics of the input into the model, if necessary. With these assumptions, the problem becomes one of identifying a state–space realization from its impulse response data, subject to a periodicity constraint, precisely the type of problem solved in section 4. Furthermore, the identification error used by the Frobenious norm discussed in section IV-B measures the mean square error between the pixels of two textured images from the same family. The potential of this approach is illustrated in Fig. 1, where it was used to expand partial images. This was accomplished by first identifying the underlying 2-D model and then simply computing its impulse response.

Fig. 1. Using 2-D Models to Expand Images

B. Texture Classification

In this section we briefly indicate how the models obtained above can be used for texture classification. Proceeding as in [17], [16], we will recast the problem into a robust model (in)validation form and exploit some recent results on semi–blind invalidation [18]. To this effect, we will postulate that all images corresponding to realizations of a given texture $\mathcal{T}$ can be obtained as the output of a 2-D operator S to an unknown input signal e with unit spectral density, applied in $(-\infty, 0] \times (-\infty, 0]$. This leads to the set-up shown in Figure 2, where $T(z_1, z_2)$ represents a nominal model of a particular texture, $h(i, j)$ and $y(i, j)$ denote the intensity value of the ideal and actual images, respectively, and where the (unknown) operator $\Delta(z_1, z_2)$ describes the mismatch between these two images.

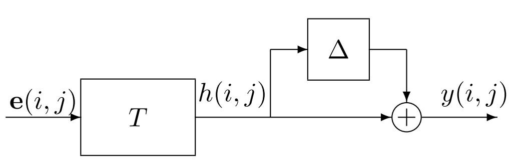

Fig. 2. The Texture Recognition Set-up

In this context, given a set of texture families, each represented by a model T_i, an unknown specimen can be classified by (i) performing a sequence of invalidation models to find the lowest uncertainty value $\|\Delta_i\|$ required to explain the specimen in terms of the model T_i, and (ii) assigning the unknown texture to the family corresponding to smallest uncertainty norm. By using the proposed identification technique to obtain a (separable) model of the nominal texture, the corresponding 2-D model invalidation problem can be reduced to two decoupled 1-D semi–blind validation problems that can be solved using the LMI–based technique developed in [18].

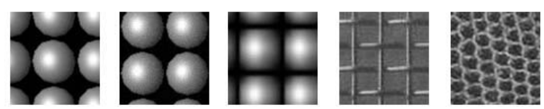

Fig. 3. The Sample Textures 1-5

TABLE I

OPTIMAL γ FOR THE TEXTURES SHOWN IN FIGURE 3

	$I_f^{1,2}$	$I_g^{1,2}$	$I_f^{1,3}$	$I_g^{1,3}$	$I_f^{1,4}$	$I_g^{1,4}$	$I_f^{1,5}$	$I_g^{1,5}$
γ_{opt}	<0.01	<0.01	0.5	0.5	0.70	0.75	0.85	0.83

Table I shows the results of applying the technique outlined above to the textures shown in Figure 3. Here $I_f^{1,j}$ and $I_g^{1,j}$ denote the results obtained when comparing the decompositions corresponding to the first image against the models obtained from the j^{th} texture. As shown in the table, the proposed technique correctly indicates that the first three images belong to the same family[1].

VI. CONCLUSIONS

Many problems of practical interest require identifying reduction of 2–D systems having a periodic impulse response. Currently available techniques are not well suited for solving these problems, since they cannot guarantee that this periodicity will be preserved, a key requirement for the applications of interest here.

[1]The higher values of $I_f^{1,3}$ and $I_v^{1,3}$ are due to the use of a lower quality image for the third texture.

Motivated by some earlier decomposition–based work on 2-D realizations and existing subspace identification methods, in this paper we address these problems by working directly with two (circulant) Hankel matrices obtained from the experimental data, via an SVD decomposition. The main result of the paper shows that the block circulant structure of these matrices can be exploited to obtain a separable in denominator state space 2-D system whose response interpolates the experimental data within the measurement noise level. These results were illustrated with several non–trivial practical examples arising in the context of textured image processing: texture synthesis and texture classification.

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VII. APPENDIX

A. *Proof of Theorem 1*

Begin by rewritting (7) in matrix form as:

$$\mathbf{H}_y = \mathbf{H}_g\mathbf{H}_u + \mathbf{H}_v \tag{15}$$

Since $\mathbf{H}_u$ is unitary and the set $\mathcal{H}$ is an algebra over the field of reals, it follows that there exists an admissible noise sequence $v(.,.) \in \mathcal{N}$ such that (15) holds if and only if there exists some matrix $\mathbf{V} \in \mathcal{H}, \|\mathbf{V}\|_F \leq \sqrt{NM}\epsilon$ such that

$$\mathbf{V} = \mathbf{H}_y\mathbf{H}_u^T - \mathbf{H}_g \doteq \mathbf{Y} - \mathbf{H}_g$$

The remainder of the proof proceeds by showing constructively that if condition (13) holds, then there exist a plant $G \in \mathcal{S}$ and an admissible noise sequence v such that (15) holds. From the results in [16] it follows that the matrices A_f and A_g satisfy the conditions $A_f^N = I,\ A_g^M = I$. This fact, combined with (2) shows that the impulse response of (14) satisfies the required periodicity constraints. Finally, by construction, (2) and (14) we have that

$$\begin{gathered} g(i,j) = C_f A_f^{i-1} B_f C_g A_g^{j-1} B_g = \hat{f}_i \cdot \hat{g}_j, \\ \hat{f}_i \doteq C_f A_f^{i-1} B_f,\ \ \hat{g}_j \doteq C_g A_g^{j-1} B_g, \end{gathered}$$

$$[g(i,j)] = \mathbf{F}_r \cdot \mathbf{G}_s = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_n \end{bmatrix} \cdot \begin{bmatrix} \hat{g}_1 & \hat{g}_2 & \cdots & \hat{g}_m \end{bmatrix},$$

where $[g(i,j)]$ denotes the impulse response matrix of the identified system. Moreover, from the circulant block structure of $\mathbf{H}_u$, $\mathbf{H}_y$ and $\mathbf{H}_g$, we have

$$\|(\mathbf{Y} - \mathbf{H}_g)\|_F = \sqrt{NM}\|(\mathbf{Y}_0 - \mathbf{F}_r\mathbf{G}_s)\|_F. \tag{16}$$

To complete the proof, we need to show that

$$\|\mathbf{Y}_0 - \mathbf{F}_r\mathbf{G}_s\|_F = \|\mathbf{F}\mathbf{G} - \mathbf{F}_r\mathbf{G}_s\|_F \leq \epsilon.$$

To this effect begin by writting:

$$\begin{aligned} \|\mathbf{F}\mathbf{G} - \mathbf{F}_r\mathbf{G}_s\|_F &\leq \|(\mathbf{F} - \mathbf{F}_r)\mathbf{G}\|_F + \|\mathbf{F}(\mathbf{G} - \mathbf{G}_s)\|_F \\ &+ \|(\mathbf{F} - \mathbf{F}_r)(\mathbf{G} - \mathbf{G}_s)\|_F \\ &\leq \bar{\sigma}(\mathbf{Y}_0)\|(\mathbf{F} - \mathbf{F}_r)\|_F + \bar{\sigma}(\mathbf{Y}_0)\|(\mathbf{G} - \mathbf{G}_r)\|_F \\ &+ \sqrt{p}\|(\mathbf{F} - \mathbf{F}_r)\|_F\|(\mathbf{G} - \mathbf{G}_s)\|_F. \end{aligned} \tag{17}$$

From the definitions of $\mathbf{F}$ and $\mathbf{F}_r$ it follows that

$$\begin{aligned} \|(\mathbf{F} - \mathbf{F}_r)\|_F &= \frac{1}{\sqrt{N}}\|(\mathbf{H}_f - \mathbf{H}_{r,f})\|_F \\ &= \left(\frac{\sum_{i=r+1}^{n_f}(\sigma_i^f)^2}{N}\right)^{1/2}. \end{aligned} \tag{18}$$

where $\mathbf{H}_{r,f} = U_{r,f} S_{r,f} V_{r,f}^T$.

Similarly,

$$\|(\mathbf{G} - \mathbf{G}_s)\|_F = \left(\frac{\sum_{i=s+1}^{n_g} (\sigma_i^g)^2}{M} \right)^{1/2}. \tag{19}$$

Combining the inequalities above yields:

$$\|\mathbf{V}\|_F \leq \sqrt{NM}(e_1 + e_2 + e_3) \leq \sqrt{NM}\epsilon.$$