

Gray Scale Structuring Element Decomposition

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Abstract

Efficient implementation of morphological operations requires the decomposition of structuring elements into the dilation of smaller structuring elements. Zhuang and Haralick presented an algorithm to find optimal decompositions of structuring elements in binary morphology. In this paper we extend the algorithm to find optimal structuring element decomposition for gray scale morphology.

1 Introduction

When the structuring element used in a morphological operation is larger than the largest element that the hardware can handle in one stage, the structuring element must be decomposed into smaller structuring elements. Each of these elements has to be such that the hardware is capable of handling it and whose morphological composition is the given structuring element.

An algorithm that finds an optimal decomposition for binary structuring elements was proposed in [1]. All binary morphological operations are naturally extended to gray scale imagery by using the Top and Umbra operations. In this paper we use this idea to extend the algorithm proposed in [1] to be used in gray scale morphology.

2 Statement of the problem

The gray scale structuring element decomposition is formally stated as:

Given a structuring element $s(x)$, find the smallest integer M and the corresponding structuring elements $h_1(x), h_2(x), \dots, h_M(x)$ such that

$$s(x) = (h_1 \oplus h_2 \oplus \dots \oplus h_M)(x)$$

where each $h_i(x)$ satisfies the hardware requirements.

In this paper we will consider that the structuring element s is a discrete function defined in a finite domain.

3 Notation and definitions

3.1 Gray scale morphology

For completeness we begin by stating some definitions and theorems. An extended presentation of the definitions and the proof of the theorems in this subsection can be found in Haralick et al. [2].

The four binary morphological operations of dilation, erosion, opening and closing are all extended to gray scale morphology by introducing the concept of the top surface of a set and the related concept of the umbra of a surface.

Def. 3.1 Let $A \subseteq E^N$ and $F = \{x \in E^{N-1} | \text{for some } y \in E, (x, y) \in A\}$. The top or top surface of A , denoted by $T[A] : F \rightarrow E$, is defined by

$$T[A](x) = \max\{y | (x, y) \in A\}.$$

Def. 3.2 A set $A \subseteq E^{N-1} \times E$ is an umbra if and only if $(x, y) \in A$ implies that $(x, z) \in A$ for every $z \leq y$.

Def. 3.3 Let $F \subseteq E^{N-1}$ and $f : F \rightarrow E$. The umbra of f , denoted by $\bar{U}[f], U[f] \subseteq F \times E$, is defined by

$$U[f] = \{(x, y) \in F \times E | y \leq f(x)\}.$$

The top of the umbra of a function f is the function f itself:

Theorem 3.1 Let $F \subseteq E^{N-1}$ and $f : F \rightarrow E$. Then $T[U[f]] = f$.

Having defined the operations of taking the Top set of a set and the Umbra of a surface, gray scale dilation and erosion can be defined:

Def. 3.4 Let $F, K \subseteq E^{N-1}$ and $f : F \rightarrow E$ and $k : K \rightarrow E$. The dilation of f by k is denoted by $f \oplus k, f \oplus k : F \oplus K \rightarrow E$, and is defined by

$$f \oplus k = T[U[f] \oplus U[k]].$$

Def. 3.5 Let $F, K \subseteq E^{N-1}$ and $f : F \rightarrow E$ and $k : K \rightarrow E$. The erosion of f by k is denoted by $f \ominus k, f \ominus k : F \ominus K \rightarrow E$, and is defined by

$$f \ominus k = T[U[f] \ominus U[k]].$$

Gray scale dilation and erosion can be accomplished by using maximum and minimum operations.

Theorem 3.2 Let $f : F \rightarrow E$ and $k : K \rightarrow E$. Then $f \oplus k : F \oplus K \rightarrow E$ can be computed by

$$(f \oplus k)(x) = \max\{f(x-z) + k(z) \mid x-z \in F \text{ and } z \in K\}.$$

Theorem 3.3 Let $f : F \rightarrow E$ and $k : K \rightarrow E$. Then $f \ominus k : F \ominus K \rightarrow E$ can be computed by

$$(f \ominus k)(x) = \min\{f(x+z) - k(z) \mid z \in K\}.$$

Gray scale opening and closing are defined in an analogous way to opening and closing in the binary morphology:

Def. 3.6 Let $f : F \rightarrow E$ and $k : K \rightarrow E$. The opening of f by the structuring element k is denoted by $f \circ k$ and is defined by

$$f \circ k = (f \ominus k) \oplus k.$$

Def. 3.7 Let $f : F \rightarrow E$ and $k : K \rightarrow E$. The closing of f by the structuring element k is denoted by $f \bullet k$ and is defined by

$$f \bullet k = (f \oplus k) \ominus k.$$

3.2 Translation in gray scale morphology

To solve the decomposition problem we need to define the translation of a gray scale structuring element. In this subsection we define the translation of a structuring element by a point and derive some useful properties.

Def. 3.8 Let $F \subseteq E^{N-1}$ and $f : F \rightarrow E$. The translation of f by (α, β) with $\alpha \in E^{N-1}$ and $\beta \in E$, is denoted by $(f)_{(\alpha, \beta)}$ and is defined by

$$(f)_{(\alpha, \beta)}(x) = f(x - \alpha) + \beta$$

for every $x \in F$.

Given $F \subseteq E^{N-1}$, $f : F \rightarrow E$ and $(\alpha, \beta) \in E^N$. The gray scale translation has the following properties.

Theorem 3.4 $(U[f])_{(\alpha, \beta)} = U[f_{(\alpha, \beta)}]$

Proof: Let $(x, y) \in E^N$. The point $(x, y) \in (U[f])_{(\alpha, \beta)}$ if and only if there exists $(u, v) \in U[f]$ such that $(x, y) = (u, v) + (\alpha, \beta)$.

By definition of umbra $(u, v) \in U[f]$ if and only if $v \leq f(u)$. Then, $y = v + \beta \leq f(u) + \beta = f(x - \alpha) + \beta$. Hence, $(x, y) \in U[f_{(\alpha, \beta)}]$. \diamond

Theorem 3.5 $T[(U[f])_{(\alpha, \beta)}] = (T[U[f]])_{(\alpha, \beta)}$

Proof: Let $x \in E^{N-1}$. Then by theorem 3.4, $T[(U[f])_{(\alpha, \beta)}](x) = T[U[f_{(\alpha, \beta)}]](x) = f_{(\alpha, \beta)}(x) = (T[U[f]])_{(\alpha, \beta)}(x)$. \diamond

The translation of a function can be expressed as the dilation of the given function and another function given by the translation point.

Theorem 3.6 There exists $G \subseteq E^{N-1}$, $g : G \rightarrow E$ such that $f_{(\alpha, \beta)} = f \oplus g$.

Proof: Let $G = \alpha$ and $g(\alpha) = \beta$. By definition 4, $f \oplus g = T[U[f] \oplus U[g]]$.

A point $(x, y) \in U[f] \oplus U[g]$ if and only if there exists $(x_1, y_1) \in U[f]$ and $(x_2, y_2) \in U[g]$ such that $(x, y) = (x_1, y_1) + (x_2, y_2)$.

But, $(x, y) \in U[f]$ implies $y \leq f(x)$ and $(x_2, y_2) \in U[g]$ implies $x_2 = \alpha$ and $y_2 \leq g(x_2) = \beta$.

Then, $y = y_1 + y_2 \leq f(x_1) + \beta = f(x - \alpha) + \beta$. Hence, by definition of Top, $(f \oplus g)(x) = \max\{z \mid z \leq f(x - \alpha) + \beta\} = f(x - \alpha) + \beta = f_{(\alpha, \beta)}$. \diamond

The dilation and erosion operations commute with translation.

Theorem 3.7 Let $f_1 : E^{N-1} \rightarrow E$ and $f_2 : E^{N-1} \rightarrow E$. Then, $f_{1(\alpha, \beta)} \oplus f_2 = (f_1 \oplus f_2)_{(\alpha, \beta)}$.

Proof: By definition of dilation and by theorem 3.5, $f_{1(\alpha, \beta)} \oplus f_2 = T[U[f_{1(\alpha, \beta)}] \oplus U[f_2]] = T[(U[f_1] \oplus U[f_2])_{(\alpha, \beta)}] = (T[U[f_1] \oplus U[f_2]])_{(\alpha, \beta)} = (f_1 \oplus f_2)_{(\alpha, \beta)}$. \diamond

Theorem 3.8 If F is finite, then $f_{(\alpha, \beta)} \ominus f = g(x)$ with $g : G = \alpha \rightarrow E$ and $g(\alpha) = \beta$.

Proof: By definition of erosion and by theorem 3.5, $f_{(\alpha, \beta)} \ominus f = T[U[f_{(\alpha, \beta)}] \ominus U[f]] = T[(U[f])_{(\alpha, \beta)} \ominus U[f]] = T[(U[f] \ominus U[f])_{(\alpha, \beta)}]$.

A point $(x, y) \in U[f] \ominus U[f]$ if and only if for every $(u, v) \in U[f]$, $(x, y) + (u, v) \in U[f]$. But, $(x, y) + (u, v) \in U[f]$ if and only if $x + u \in F$ and $y + v \leq f(x + u)$.

If F is finite, then there exists X_1 and X_2 such that $X_1 \leq x \leq X_2$ for every $x \in F$.

Let $u = X_1$. Then, $x + X_1 \in F$ implies $x \geq 0$. Now let $u = X_2$. Then, $u + X_2 \in F$ implies $x \leq 0$. Hence, x must be 0.

Consider $(u, v) = (X_1, f(X_1)) \in U[f]$. But, $y + v = y + f(X_1) \leq f(X_1) \Rightarrow y \leq 0$.

Now, $f_{(\alpha, \beta)} \ominus f = T[\{(x, y) \mid x = 0, y \leq 0\}]_{(\alpha, \beta)} = T[\{(x, y) \mid x = \alpha, y \leq \beta\}] = (\alpha, \beta)$. \diamond

4 Reduction of the problem.

The problem we solve in this section is to construct a decomposition of s , into h_1, h_2, \dots, h_M , having the smallest M , if one exists, where each h_i has no more points than the prescribed fixed number k determined by hardware constraints. We also assume that the domain of s is finite.

To determine such a decomposition of s , if one exists, requires a combinatorial search process for the domains of the functions h_i and for the values of these functions at each point of the domain. Our algorithm limits the possible domains and the possible values of the functions h_i , and thereby greatly reduces the search space.

We will show that the gray scale structuring element decomposition is equivalent to one with fewer degrees of freedom:

Theorem 4.1 If $s = h_1 \oplus h_2 \oplus \dots \oplus h_M$, then there exists (α, β) and j_1, j_2, \dots, j_M such that:

1. $\#j_i = \#h_i$, for $i = 1, \dots, M$
2. $j_i(0) = 0$ and $j_i(x) \leq 0$, for $i = 1, \dots, M$
3. $s = \{(\alpha, \beta)\} \oplus j_1 \oplus j_2 \oplus \dots \oplus j_M$
4. $\{(\alpha, \beta)\} = s \ominus \{j_1 \oplus j_2 \oplus \dots \oplus j_M\}$

Proof: Let α_i be such that $h_i(\alpha_i) = \beta_i$ and $h_i(x) \leq \beta_i$ for every x in the domain of h_i .

Consider $j_i = (h_i)_{(-\alpha_i, -\beta_i)}$. Then, $j_i(0) = h_i(\alpha_i) - \beta_i = 0$ and $j_i(x) \leq 0$ for every x in the domain of j_i . Note that with this definition h_i and j_i have the same number of points.

Next we will show that $s = \{(\alpha, \beta)\} \oplus j_1 \oplus j_2 \oplus \dots \oplus j_M$. Since $j_i = (h_i)_{(-\alpha_i, -\beta_i)}$, by the definition of translation, we have that $h_i = (j_i)_{(\alpha_i, \beta_i)}$. Therefore,

$$s = h_1 \oplus h_2 \oplus \dots \oplus h_M = ((j_1)_{(\alpha_1, \beta_1)} \oplus \dots \oplus (j_M)_{(\alpha_M, \beta_M)})$$

By theorem 3.7,

$$s = (j_1 \oplus j_2 \oplus \dots \oplus j_M)_{(\alpha, \beta)}$$

where

$$(\alpha, \beta) = (\alpha_1 + \alpha_2 + \dots + \alpha_M, \beta_1 + \beta_2 + \dots + \beta_M)$$

Then, by theorem 3.6, s can be expressed as the dilation $s = \{(\alpha, \beta)\} \oplus (j_1 \oplus j_2 \oplus \dots \oplus j_M)$. Furthermore, by theorem 3.8 $\{(\alpha, \beta)\} = s \ominus (j_1 \oplus j_2 \oplus \dots \oplus j_M)$. \diamond

Theorem 4.1 in effect shows that the original problem can be reduced to one with fewer degrees of freedom. Since $j_i(0) = 0$, there is one less unknown point to determine each j_i compared to the number of unknown points in the corresponding h_i . Furthermore, the extra unknown (α, β) is determined without searching, once all the j_i have been determined.

The search for the decomposition elements j_i can be reduced by establishing that if $s = \{(\alpha, \beta)\} \oplus j_1 \oplus j_2 \oplus \dots \oplus j_M$, with $j_i(0) = 0$ and $j_i(x) \leq 0$, then there exists x_1 and x_2 in the domain of s such that $x = x_1 - x_2$ and $s(x_1) - s(x_2) \leq j_i(x) \leq 0$. The proof consists of various steps given by theorems 4.2 to 4.6.

Theorem 4.2 Let $f_1 : F_1 \rightarrow E$ and $f_2 : F_2 \rightarrow E$. Then, for every $x \in F_1 \oplus F_2$ there exists $x_1 \in F_1$ and $x_2 \in F_2$ such that $x = x_1 + x_2$ and $(f_1 \oplus f_2)(x) = f_1(x_1) + f_2(x_2)$.

Proof: By theorem 3.2, $(f_1 \oplus f_2)(x) = \max\{f_1(x - u) + f_2(u) \mid x - u \in F_1 \text{ and } u \in F_2\}$.

Let u_1 be such that $f_1(x - u_1) + f_2(u_1)$ is maximum. Then there exists $x_1 = x - u_1 \in F_1$ and $x_2 = u_1 \in F_2$ such that $x = x_1 + x_2$ and $(f_1 \oplus f_2)(x) = f_1(x_1) + f_2(x_2)$. \diamond

Theorem 4.3 Let $A \subseteq E^{N-1}$ and $B \subseteq E^{N-1}$. Then, $A \circ B \subseteq A$.

Proof: Let $a \in A \circ B$. Then there exists an $x \in A \oplus B$ and $b \in B$ such that $a = x + b$. But $x \in A \oplus B$ implies that for every $y \in B$, $x + y \in A$. Since $b \in B$, $x + b \in A$. But $a = x + b$ so that $a \in A$. \diamond

Theorem 4.4 Let $A = B \oplus C$. Then, $A \circ C = A$.

Proof: Let $a \in A$. Since $A = B \oplus C$, there exists a $b \in B$ and a $c \in C$ such that $a = b + c$. But for every $x \in B$ and $c \in C$, $x + c \in A$. Since $b \in B$, we must have for every $y \in C$, $b + y \in A$. This implies $b \in A \oplus C$ and $b \in B$ imply $a \in (A \oplus C) \oplus C$. Hence $A \subseteq (A \circ C)$. By theorem 4.3, $A \circ C \subseteq A$. Thus, $A \circ C = A$. \diamond

Theorem 4.5 Let $s = a \oplus j$. Then, $s = s \circ j$.

Proof: By definition of dilation, $s(x) = T[U[a] \oplus U[j]](x)$. Taking the umbra at both sides, $U[s] = U[a] \oplus U[j]$. By theorem 4.4, $U[s] = U[s] \circ U[j]$. Then, $s(x) = T[U[s] \circ U[j]](x) = (s \circ j)(x)$. \diamond

Theorem 4.6 Let $s = s \circ j$, $j(0) = 0$ and $j(x) \leq 0$. Then, there exists x_1 and x_2 in the domain of s such that $x = x_1 - x_2$ and $j(x) \geq s(x_1) - s(x_2)$.

Proof: Since $s = (s \circ j) \oplus j$, by theorem 4.2, for every x in the domain of j there exists x_1 and x_2 in the domain of s such that $x = x_1 - x_2$ and $s(x_1) = (s \circ j)(x_2) + j(x)$. But by theorem 3.3, $(s \circ j)(y) = \min\{s(y + u) - j(u) \mid u \in J\} \leq s(y)$ for every $y \in S \circ J$, since $j(0) = 0$ and $j(x) \leq 0$. Then, $j(x) \geq s(x_1) - s(x_2)$ and $x = x_1 - x_2$. \diamond

We have shown that the possible domains of the functions j_i are those formed by differences between points of the domain of s . Furthermore, we showed that $s(x_1) - s(x_2) \leq j_i(x_1 - x_2) \leq 0$ for each point.

4.1 Search.

In this section we give a complete description of the tree search using the elements developed in the previous section. The algorithm is a generalization of the algorithm proposed in [1].

Suppose that the structuring element s has m points and that each j_i must have k points one of which is $(0, 0)$.

To reduce the tree search we take into account the following two considerations:

I) Since dilation is commutative, for a node associated with j_q there is only need to consider the possible descendents $j_{q+1}, j_{q+2}, \dots, j_J$.

II) Forward checking can be used to control the growth of the tree. We will show that if at some level l , j_1, j_2, \dots, j_l have been determined, then the only j 's that need to be considered for any node in the subtree below are those that satisfy:

$$(t_l \ominus j) \oplus j \oplus k_l = s$$

where $t_l = s \ominus k_l$, and $k_l = j_1 \oplus j_2 \oplus \dots \oplus j_l$.

Assume that $s = \{(\alpha, \beta)\} \oplus j_1 \oplus \dots \oplus j_M$ with $M > l$, and we need to find j_{l+1} . The structuring element s can be rewritten as:

$$s = (\{(\alpha, \beta)\} \oplus j_{l+2} \oplus \dots \oplus j_M) \oplus k_l \oplus j_{l+1}$$

Then, by theorem 4.5, s must be open under k_l dilated by the function j , the candidate to be selected as j_{l+1} :

$$s = (s \ominus (k_l \oplus j)) \oplus (k_l \oplus j) = (t_l \ominus j) \oplus j \oplus k_l$$

It is also true, that this condition holds for the level l considered and for any node in the subtree below: Let $m > l$. Then, j to be considered as a child of a node at level m must satisfy: $s = s \circ (k_m \oplus j)$, but, $k_m \oplus j = (k_l \oplus j) \oplus (j_{l+1} \oplus j_{l+2} \oplus \dots \oplus j_M)$.

Theorem 4.7 If $S = S \circ (A \oplus B)$ then $S = S \circ A$.

Proof: See Haralick et al. [2].

Theorem 4.8 If $s = s \circ (a \oplus b)$ then $s = s \circ a$.

Proof: By hypothesis, $s(x) = T[U[s] \circ U[a \oplus b]](x)$. Then, by theorem 4.7, $U[s] = U[s] \circ (U[a] \oplus U[b]) = U[s] \circ U[a]$. Therefore, $s(x) = T[U[s] \circ U[a](x)$ and $s(x) = (s \circ a)(x)$. \square

Theorem 4.8 shows that if s is not open under $k_l \oplus j$, then it is not open under $k_m \oplus j$. This condition can be used as a forward check to prune the search tree.

4.2 Algorithm

The algorithm to accomplish the decomposition of a gray scale structuring element s consists of a breadth first tree search with forward checking.

A node i in the tree corresponds to a candidate structuring element, $j_i(x)$. Each node i has also associated with it the following entities:

1. A list of all its possible descendents, L_i .
2. The partial decomposition so far, $k_i = j_1 \oplus j_2 \dots \oplus j_i$, corresponding to its dilation with all its predecessor nodes.
3. The undecomposed part or residue, $t_i = s \ominus k_i$.

The root of the tree is initialized such that $L_0 = j_1, j_2, \dots, j_M$ is the set of all possible structuring elements, $k_0 = \{0\}$, and $t_0 = s$. Before opening a node in the tree a forward check is made through the possible descendents of the node. The checking eliminates those structuring elements in L_i that do not satisfy $s = (t_i \ominus j) \oplus (j \oplus k_i)$. Once the forward checking is finished, the nodes corresponding to the elements in L_i that survived the test are opened.

Any decomposition found at the lower level of the tree is an optimal decomposition. A decomposition is found when the number of elements in the domain of the residue t is one, and its functional value $t(\alpha) = \beta$ corresponds to the (α, β) appearing in the decomposition $s = \{(\alpha, \beta)\} \oplus j_1 \oplus \dots \oplus j_M$.

5 Example

In the following we give an example illustrating the use of the algorithm to decompose an structuring element into sets of two points. Let $s(x)$ be the gray scale structuring element to be decomposed, as shown in figure 1. There are thirty five candidate structuring elements, defined by $j_i(0) = 0; j_i(x_i) = y$, with $0 \leq y \leq s(x_1) - s(x_2)$, and $x_i = x_1 - x_2$. However,

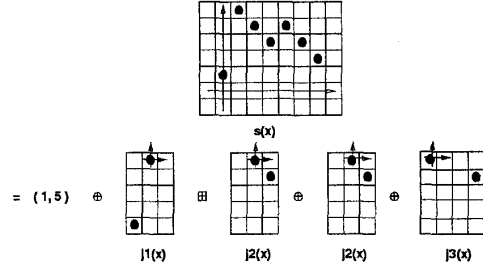


Figure 1: A decomposition of a gray scale structuring element.

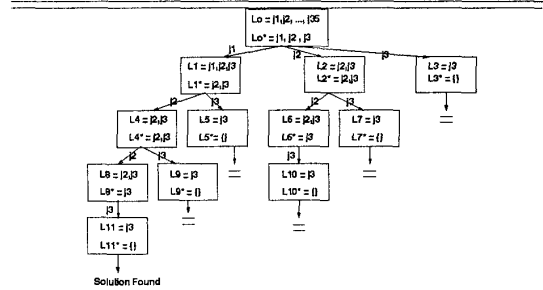


Figure 2: Illustrates the tree search for $s(x)$.

only three of the candidates survive the forward check: $\{j_1(0) = 0; j_1(-1) = -4\}$, $\{j_2(0) = 0; j_2(1) = -1\}$, and $\{j_3(0) = 0; j_3(3) = -1\}$. Figure 2 shows the tree created by the algorithm. The optimal decomposition of s , $s = \{(1, 5)\} \oplus j_1(x) \oplus j_2 \oplus j_3$ is shown in figure 1.

6 Conclusion

The gray scale structuring element decomposition problem can be solved in a similar way to the binary structuring element decomposition problem. In this paper, we showed that the decomposition problem can be solved by simply searching among a finite set of values. The essence of the algorithm is the same as the essence of the binary problem: (1) the domain of the structuring elements participating in the decomposition must have members which are the differences between members of the domain of the given structuring element and (2) that it is necessary for the undecomposed part of the structuring element to be morphologically open with respect to any structuring element participating in its further decomposition.

7 References

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