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A convex optimization approach to fixed–order controller design for disturbance rejection in SISO systems.

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Abstract

The problem of rejection of persistent unknown–but–bounded disturbances can be solved using the well–known l^1 design approach. However, in spite of its success, this theory suffers from the fact that the resulting controller may have arbitrarily high order, even in the state–feedback case. In addition, system performance is optimized under the assumption of zero initial conditions. In this paper we propose a new approach to the problem of synthesizing fixed order controllers to optimally reject persistent disturbances. The main result of the paper shows that this approach leads to a finite–dimensional convex optimization problem that can be efficiently solved.

1 Introduction

A large number of control problems can be recast as the problem of synthesizing a controller capable of stabilizing a given linear time invariant system while, at the same time, minimizing the worst case response to some exogenous disturbances. When the signals involved are persistent bounded signals, with size measured in terms of peak time–domain values, it leads to l^1 optimal control theory [9, 3, 4, 6] (see also [1] for earlier related work).

The l^1 theory success lies on the fact that it directly incorporates time–domain specifications. Moreover, it furnishes a complete solution to the robust performance problem [5]. However, in contrast with $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control, l^1 optimal controllers can have arbitrarily high order [7]. Moreover, this theory cannot accommodate non–zero initial conditions.

Motivated by these difficulties, in this paper we propose a new approach to synthesizing fixed order controllers for persistent disturbance rejection in SISO systems. This approach is based upon considering an expanded class of problems that includes ℓ^1 theory as a limit case (when the order of the controller is free).

The basic idea of the paper is the concept of *equalized performance*. In plain words a a linear SISO plant of order n achieves an equalized performance level μ if, whenever n consecutive output values have magnitude less than μ, the same condition is repeated in the future. Thus having finite equalized performance is a stronger property than stability (while having finite ℓ^1 induced norm is equivalent to asymptotic stability). Nevertheless, as we show in the sequel, finite equalized performance can be achieved by closing the loop with a controller having at least the same order of the plant.

The main results of the paper can be summarized as follows.

- the problem of finding a *fixed order* controller achieving a given equalized performance level μ leads to a linear programming problem whose dimension is know a priori and it does not depend on the problem data.
- The optimal value of μ (and the corresponding controller) can be computed in polynomial time.
- The proposed technique is applicable even in cases where ℓ^1 theory breaks down, such as when the plant has zeros on the stability boundary.

For brevity, some of the results are presented without proofs for which the reader is referred to the full version of the paper [2].

2 The equalized performance problem

2.1 Notation

Given a sequence $h \in \ell^1$, its λ–transform is defined as $H(\lambda) \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} h_i \lambda^i$ [1]. Given a polynomial $P(\lambda) = \sum_{i=0}^{n} a_i \lambda^i$ we denote its coefficients vector as $a \stackrel{\text{def}}{=} [a_o \ \ a_1 \ \ \dots \ \ a_{n-1}]^T$. The vector a deprived of the leading coefficient will be denoted by $\tilde{a}$, i.e. $\tilde{a} \stackrel{\text{def}}{=} [a_1 \ \ \dots \ \ a_{n-1}]^T$.

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[1] Note that this is the inverse of the usual z transform. Therefore for causal, stable systems $H(\lambda)$ is analytical in $|\lambda| < 1$.

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2.2 Definitions and Preliminary Results

Consider a stable SISO plant defined by the following transfer function:

$$e(\lambda) = \frac{\sum_{j=0}^{n} b_j \lambda^j}{\sum_{i=0}^{n} a_i \lambda^i} w(\lambda), \quad a_0 = 1. \tag{1}$$

To this plant we can associate the following ARMA model

$$e(k) = -\sum_{i=1}^{n} a_i e(k-i) + \sum_{j=0}^{n} b_j w(k-j) \tag{2}$$

or equivalently, the set of equations:

$$\begin{aligned} x_1(k+1) &= -a_1 x_1(k) + x_2(k) + (b_1 - b_0 a_1) w(k) \\ x_2(k+1) &= -a_2 x_1(k) + x_3(k) + (b_2 - b_0 a_2) w(k) \\ &\vdots \\ x_n(k+1) &= -a_n x_1(k) + (b_n - b_0 a_n) w(k) \\ e(k) &= x_1(k) + b_0 w(k) \end{aligned} \tag{3}$$

For any positive integer k we have that

$$e^{(k)} = \mathcal{O}^{(k)} x(0) + \mathcal{H}^{(k)} w^{(k)} \tag{4}$$

where

$$\begin{aligned} w^{(k)} &\stackrel{\text{def}}{=} [w(0) \quad w(1) \quad \dots \quad w(k-1)]^T \\ e^{(k)} &\stackrel{\text{def}}{=} [e(0) \quad e(1) \quad \dots \quad e(k-1)]^T \\ \mathcal{H}^{(k)} &\stackrel{\text{def}}{=} \begin{bmatrix} b_0 & 0 & 0 & \dots & 0 \\ CB & b_0 & 0 & \dots & 0 \\ CAB & CB & b_0 & \dots & 0 \\ \dots & \dots & & \dots & 0 \\ CA^{k-2}B & \dots & \dots & CB & b_0 \end{bmatrix} \\ \mathcal{O}^{(k)} &\stackrel{\text{def}}{=} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{k-1} \end{bmatrix} \end{aligned} \tag{5}$$

Considering the above relationship in the case $k = n$ establishes a (well-known) correspondence between any minimal quadruple (A, B, C, D) and the ARMA model (2), in the following sense: Given any $w(k)$ and any initial condition $x(0)$, the corresponding ouput sequence $e(k)$ of the former is an admissible evolution of the latter. Conversely, any evolution of the ARMA model is an admissible output sequence for the system having the state space realization (3), for a suitable choice of the initial state $x(0)$. Since $\mathcal{O}^{(n)}$ is invertible (recall that (A, B, C, D) is minimal), determining $x(0)$ is immediate. Also note that *for a given sequence $w(k)$, there is a one to one correspondence between the first n values of $e(k)$ and the initial condition $x(0)$.*

Next we recall the usual ℓ^1 performance definition:

Definition 1 *The plant (1) has ℓ^1 performance less than μ_{ℓ^1} iff for $x_i(0) = 0$, $i = 1, \dots n$, and for all sequences $w(k)$, $k = 0, 1, \dots$, such that $|w(k)| \le 1$, we have $|e(k)| \le \mu_{\ell^1}$.*

Motivated by this definition, we introduce now the concept of *equalized performance*.

Definition 2 *A stable plant of the form (1) has (finite) equalized performance less than μ iff for $|e(i)| \le \mu$, $i = 0, \dots n-1$, and for $|w(j)| \le 1$, $j = 0, 1, \dots$,*

$$|e(k)| \le \mu, \quad k \ge n \tag{6}$$

The term *equalized* stems from the fact that the definition above is strictly equivalent to setting the first n values of $|e(k)|$ *all equal to* μ (in all possible ways) and requiring that $|e(k)| \le \mu$ in the future.

So far we have considered the case where the length of the output string coincides with the McMillan degree of the plant (in the sequel we will sometimes refer to this case as the *natural* performance case). However, addressing some technical points such as stable pole/zero cancellations requires extending this definition to strings of length $N > n$.

Definition 3 *A stable plant of the form (1) has (finite) equalized N-performance less than μ iff for $|e(i)| \le \mu$, $i = 0, \dots N-1$, and all sequences $w(k)$, $k = 0, 1, \dots$, $|w(k)| \le 1$, compatible with $e(i)$, $i = 0, \dots, N-1$*[2] *we have that*

$$|e(k)| \le \mu, \quad k \ge N \tag{7}$$

Thus a plant achieves equalized N–performance less than μ if whenever a string of N consecutive output values $e(0), e(1), \dots, e(N-1)$ is below the magnitude μ, then the same condition is repeated in the future, for all possible values of the exogenous disturbance w *that could have generated* the sequence of output values for some appropriate initial condition. In the special case where $n = N$, the sequences $|e(k)| \le \mu$ and $|w(k)| \le 1$, $k = 0, 1, \dots, n-1$ can be chosen independently. On the other hand, if $N > n$, then the constraint $|e(i)| \le \mu$, $i = 0, \dots N-1$ *imposes an additional constraint* on the first N values of the sequence $w(k)$.

The set of admissible initial conditions, i.e. the set of initial conditions that, together with an appropriately chosen sequence of disturbances, generate a sequence of N outputs having magnitude less than μ, is given by:

[2] in the sense that there exists an initial condition $x(0)$ such that the output corresponding to this initial condition and the sequence of inputs $w(k), k = 0, \dots, N-1$ is precisely $e(i), i = 0, \dots, N-1$.

$$\mathcal{X}^{(N)}(\mu) = \{x(0): \quad \|\mathcal{O}^{(N)}x(0) + \mathcal{H}^{(N)}w^{(N)}\|_\infty \le \mu, \text{ for some } \|w^{(N)}\|_\infty \le 1\}. \tag{8}$$

Remark 1 *The set $\mathcal{X}^{(N)}(\mu)$ always contains the origin. If (A, C) is observable, it is a compact polyhedron (because the $w(k)$ are bounded). Furthermore, $\mathcal{X}^{(N')}(\mu) \subset \mathcal{X}^{(N)}(\mu)$, if $N' > N$.*

Lemma 1 *If a plant has equalized N-performance less than μ, it also has equalized N-performance less than μ' for all $\mu' \ge \mu$.*

Proof: Follows immediately from linearity, by scaling. □

This lemma implies than once N consecutive output values are below a given level $\mu' \ge \mu$, then $|e(k)| \le \mu'$ for all k. Thus we can introduce the following definition.

Definition 4 *The equalized N performance level μ^N of a stable plant is defined as: $\mu^N = \inf\{\mu :$ the plant has equalized N-performance less or equal than $\mu\}$.*

Remark 2 *It is easy to show that not all stable plants have finite equalized N–performance for a given N. However, as we show in section 3, any stable plant achieves equalized N–performance for some $\mu > 0$ provided that N is sufficiently large.*

Remark 3 *Since the set $\mathcal{X}^{(N)}(\mu)$ includes the origin, it follows that $\mu_{\ell^1} \le \mu^N$. In the special case where $\mu_{\ell^1} = \mu^N$ the plant is said to be N-equalized*

3 Equalized performance characterization

In this section we present some properties of plants achieving a given equalized N–performance level μ. Here we use the ARMA model (2) and we assume that $\|b\| \ne 0$ to avoid critical cases (the case $\|b\| = 0$ will be reconsidered later).

Theorem 1 *If the plant (1) has equalized N-performance ($N \ge n$) less than μ then it has equalized N'-performance less than μ for all $N' > N$.*

Proof. If follows immediately from the fact that $\mathcal{X}^{(N')}(\mu) \subset \mathcal{X}^{(N)}(\mu)$.

Next we address the issue of computing the equalized N–performance level of a given plant.

Theorem 2 *Let $\mu \ge 0$. The plant (1) has equalized n–performance less than μ iff the following condition holds:*

$$\mu\|\tilde{a}\|_1 + \|b\|_1 \le \mu \tag{9}$$

Therefore the equalized n-performance level μ^n of the plant is given by

$$\mu^n = \frac{\|b\|_1}{1 - \|\tilde{a}\|_1}. \tag{10}$$

Remark 4 *From Theorem 2 we have that if $b \ne 0$, a necessary condition for a plant to have finite n equalized performance is $\|\tilde{a}\|_1 < 1$. It is clear that this condition implies system stability. If $b = 0$, a necessary condition is $\|\tilde{a}\|_1 \le 1$.*

We consider now the general case where $N \ge n$. To this effect define $m = N - n$ and consider the following set of $m + 1$ equations

$$\begin{array}{rcl} e(n) &=& \sum_{i=1}^{n} a_i e(n-i) + \sum_{j=0}^{n} b_j w(n-i), \\ e(n+1) &=& \sum_{i=1}^{n} a_i e(n+1-i) + \sum_{j=0}^{n} b_j w(n+1-i), \\ &\vdots& \\ e(N) &=& \sum_{i=1}^{n} a_i e(N-i) + \sum_{j=0}^{n} b_j w(N-i) \end{array} \tag{11}$$

Eliminating $e(N-1), e(N-2), \ldots, e(n)$, yields:

$$e(N) = \sum_{i=1}^{n} a_i^{(m)} e(n-i) + \sum_{j=0}^{N} b_j^{(m)} w(N-j) \tag{12}$$

where the $a_i^{(m)}$, $i = 1, 2, \ldots, n$, and the $b_j^{(m)}$, $j = 0, 1, \ldots, N$, are functions of the coefficients a_i and b_j of (2). This expression, combined with Definition 3, leads to the following result:

Theorem 3
The plant (1) has equalized N-performance less than μ if:

$$\mu\|\tilde{a}^{(m)}\|_1 + \|b^{(m)}\|_1 \le \mu \tag{13}$$

where

$$\tilde{a}^{(m)} = \left[a_1^{(m)}, \ldots, a_n^{(m)}\right]^T, \quad b^{(m)} = \left[b_0^{(m)}, \ldots, b_n^{(m)}\right]^T.$$

Therefore an upper bound for the equalized N-performance level of an n^{th}-order plant is given by

$$\bar{\mu}^N = \frac{\|b^{(m)}\|_1}{1 - \|\tilde{a}^{(m)}\|_1}. \tag{14}$$

We stress the fact that this condition is only sufficient. Note that, contrary to the case where $N = n$, here necessity fails because now the sequences $e(k)$ and $w(k)$ cannot be chosen independently. Note that stability of the plant implies that as $m \to \infty$ then the coefficients $a_i^{(m)} \to 0$. This leads to the following important facts:

Corollary 1 *If the plant (1) is stable, it has finite equalized N-performance less than μ for some $N = n + m$, with m sufficiently large.*

Corollary 2 *If the plant (1) has a finite impulse response then it is N-equalized for all $N \geq n$.*

Next we establish that as N increases the equalized N-performance level μ^N approaches from above the ℓ^1 performance level.

Theorem 4
Let $\bar{\mu}^\infty \doteq \lim_{N\to\infty} \bar{\mu}^N = \lim_{m\to\infty} \bar{\mu}^{n+m}$. If the plant (1) is stable, then its N-equalized performance μ^N level approaches its ℓ^1 performance level μ_{ℓ^1} as $N \to \infty$, i.e.

$$\bar{\mu}^\infty = \lim_{m\to\infty} \sum_{j=0}^{n+m} |b_j^{(m)}| = \mu_{l^1} \tag{15}$$

Finally, we address the issue of equalized performance in the case where the plant realization is non-minimal. This is important in the context of synthesis because even if we start from a minimal realization, stable pole/zero cancellations may appear in the resulting closed loop system.

Theorem 5 *Consider any arbitrary monic polynomial $C(\lambda)$ and assume that the ARMA model $C(\lambda)A(\lambda)y(\lambda) = C(\lambda)B(\lambda)w(\lambda)$ of order N has equalized N-performance less than μ. Then the ARMA model $A(z)y(z) = B(z)w(z)$ also has equalized N-performance less than μ.*

Proof: The proof follows from the fact that $C(\lambda)A(\lambda)y(\lambda) = C(\lambda)B(\lambda)w(\lambda)$ corresponds to the ARMA model obtained by combining the equations in (11) using the coefficients of $C(\lambda)$.

4 Optimization of the equalized performance

In this section, we consider the problem of synthesizing fixed order controllers such that the resulting closed-loop optimally rejects (in the equalized performance sense) persistent disturbances. Consider a SISO plant of the form:

$$\begin{bmatrix} e(\lambda) \\ y(\lambda) \end{bmatrix} = G(\lambda) \begin{bmatrix} w(\lambda) \\ u(\lambda) \end{bmatrix} \tag{16}$$

where

$$G(\lambda) = \frac{1}{d(\lambda)} \left[\begin{array}{c|c} n_{11}(\lambda) & n_{12}(\lambda) \\ \hline n_{21}(\lambda) & n_{22}(\lambda) \end{array} \right] \tag{17}$$

where u, w, y and e represent the control input, exogenous disturbances, measurements available to the controller and performance output respectively. Then the optimal equalized performance problem can be precisely stated as:

Problem 1 *Given the SISO linear time-invariant plant (17) with McMillan degree r, find a linear time-invariant compensator of a given order $s \geq r$ such that the equalized n performance of the resulting closed-loop system is minimized, where $n = s + r$.*

In the sequel we show that this problem reduces to a finite-dimensional convex optimization problem. To this effect consider a controller of the form:

$$u(\lambda) = \frac{q(\lambda)}{p(\lambda)} y(\lambda), \tag{18}$$

where p is a monic polynomial of degree s. The corresponding closed-loop system is:

$$e(\lambda) = \frac{n_{11}(\lambda)}{d(\lambda)} + \frac{1}{d(\lambda)} \frac{n_{12}(\lambda)q(\lambda)n_{21}(\lambda)}{[d(\lambda)p(\lambda) - n_{22}(\lambda)q(\lambda)]} w(\lambda). \tag{19}$$

where $d(s)$ is the characteristic polynomial of A. The polynomial $[n_{11}n_{22} - n_{12}n_{21}]$ has d as a factor, i.e.

$$n_{11}(\lambda)n_{22}(\lambda) - n_{12}(\lambda)n_{21}(\lambda) = d(\lambda)\bar{n}(\lambda)$$

thus

$$\begin{aligned} [d(\lambda)p(\lambda) - n_{22}(\lambda)q(\lambda)]e(\lambda) &= [p(\lambda)n_{11}(\lambda) \\ &- q(\lambda)\bar{n}(\lambda)]w(\lambda). \end{aligned} \tag{20}$$

This last expression can be rewritten as:

$$d_{cl}(p,q)(\lambda)e(\lambda) = n_{cl}(p,q)(\lambda)w(\lambda). \tag{21}$$

Without loss of generality (by using an appropriate scaling if necessary), $p(\lambda)$ and $q(\lambda)$ can always be selected such that *the polynomial $d_{cl}(p,q)(\lambda)$ has its independent term equal to one*, that is

$$d_{cl}(p,q)(\lambda) = 1 + d_{cl,1}\lambda + d_{cl,2}\lambda^2 + \ldots. \tag{22}$$

This additional equality constraint guarantees both that the resulting loop is well-posed and that it has McMillan degree $n = s + r$.

From Theorem 2 it follows that the closed loop system (21) achieves equalized performance $\leq \mu$, if and only if

$$\mu\|\tilde{d}_{cl}(p,q)\|_1 + \|n_{cl}(p,q)\|_1 \leq \mu. \tag{23}$$

Since $n_{cl}(p,q)(\lambda)$ and $\tilde{d}_{cl}(p,q)(\lambda)$ are affine functions of the coefficients of the polynomials $p(\lambda)$ and $q(\lambda)$, and since the additional constraint (22) is equivalent to a linear constraint involving only the leading coefficients of q and p, it follows that synthesizing a controller achieving a fixed, given performance level $\mu > 0$ is equivalent to finding an interior point in a convex set in the combined p, q space. Moreover, denoting by

$$\theta \doteq \begin{bmatrix} p \\ q \end{bmatrix},$$

condition (23) above can be written as follows

$$\mu\|\Phi\theta\|_1 + \|\Psi\theta\|_1 \leq \mu. \tag{24}$$

where Φ and Ψ are suitable matrices whose entries are functions of the plant coefficients. Thus for each candidate μ, the problem of synthesizing a controller that achieves equalized performance less than μ (or establishing that none exists) reduces to solving a feasibility problem that can be recast into a Linear Programming form. If s, the order of the controller, is chosen to be at least as large as r, the order of the plant, this LP problem is always feasible for some μ large enough. This follows from the fact that in this case p and q can be chosen so that the corresponding closed–loop is a FIR, and thus (Corollary 2) has finite equalized performance. These results are summarized in the next Theorem, stating the main result of the paper.

Theorem 6 *Consider a system of the form (16) with McMillan degree r. Then for each $s \geq r$ there exists a compensator of the form (18) such that the resulting closed-loop system has finite equalized $(r+s)$–performance. Furthermore, given s, the problem of synthesizing a controller of order s that minimizes the equalized performance level can be solved by a globally converging procedure, entailing only the solution of a sequence of LP problems, each one having $6n+7$ variables, $4n+5$ inequality and $4n+5$ equality constraints.*

Remark 5 *Since both the number of constraints and variables are affine functions of n, it follows that synthesizing a controller that achieves a given equalized performance level can be solved in polynomial time. Thus, computing the optimal equalized level (within a given tolerance) can also be accomplished in polynomial time.*

Note that the synthesis algorithm proposed in Theorem 6 works even if the order of the controller is selected to be smaller than r, the order of the plant. However, in this case there is no a–priori guarantee that the problem will be feasible, even for a sufficiently large value of μ. From a practical point of view, the initial value of the controller order s_o should be selected equal to ther order of the plant. This guarantees that the parametric problem will have a solution for some $\tilde{\mu}$. Once the optimal value of the equalized performance is established for this case, we can proceed, if necessary, to decrease the order of the controller as needed. This leads to a non–increasing sequence $\mu_{opt}^{s_i} > 0$. As we state next, this sequence converges to the optimal ℓ^1 cost.

Theorem 7 *Consider an increasing sequence $s_i \geq r$ and let μ_i denote the optimal equalized performance level achievable with a controller of order s_i. Assume that the plant satisfies the standard assumptions of ℓ^1 theory and let μ_{ℓ^1} denote the optimal achievable ℓ^1 performance level. Then $\mu_i \rightarrow \mu_{\ell^1}$. Moreover, there exists $\tilde{s}$ such that $\mu_i = \mu_{\ell^1}$ for all $s_i \geq \tilde{s}$.*

5 Example

Consider the following system, taken from [7]:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{ccc|cc} 2.7 & -23.5 & 4.6 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1 & -2.5 & 1.501 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

The optimal ℓ^1 controller has order 16. The corresponding closed closed–loop is an 18^{th} order FIR, with ℓ^1 norm $\mu_{\ell^1} = 3.01$. Table 1 shows a comparison of this optimal ℓ^1 controller versus the optimal equalized controllers obtained by selecting different values for the controller order. In this particular example in all cases the resulting equalized controllers rendered the closed–loop system an FIR, and thus $\mu_{eq} = \mu_{\ell^1}$. Notice that by the time the order of the controller is selected to be 8, the corresponding performance is 3.07. Thus, when compared with the optimal ℓ^1 controller we have a significant order reduction (50%) at the price of about 2% increase in cost. Note that in this case the optimal

controller order	$\|\|\Phi\|\|_{\ell^1}$
16 (optimal ℓ^1)	3.01
3	3.85
4	3.42
6	3.16
8	3.07

Table 1: Closed–loop ℓ^1 norm for different equalized designs.

equalized closed–loop system has a finite impulse response. This raises the question of whether or not this is a general property of the method (as in the classical ℓ^1 case). Numerical experiment show that in practice the optimal equalized plant "tends to be an FIR". However, there are some counterexamples where this property does not hold.

6 Discussion of the method

In this section we comment on some of the features of the proposed method. In particular:

1) Recall that in Section 3 we assumed that $b \neq 0$. Through Theorem 2 this guarantees that $||\tilde{a}|| < 1$ which implies asymptotic stability. If $b = 0$, the inequality $\mu||\tilde{a}|| \leq \mu$ requires that $||\tilde{a}|| \leq 1$, and this property implies only marginal stability. Thus there might be trajectories that do not converge (but that do not diverge as well). Clearly, the feasible solutions p, q of (23)

might render $n_{cl}(p,q)=0$. The solution to this problem is immediate. Take ϵ arbitrarily small and replace condition (23) by the following condition

$$\mu\|\tilde{d}_{cl}(p,q)\|_1+\|n_{cl}(p,q)\|_1\le\mu-\epsilon. \qquad (25)$$

Thus if $\|n_{cl}(p,q)\|_1=0$ we still have $\|\tilde{d}_{cl}(p,q)\|_1\le 1-\epsilon$, and *asymptotic stability is guaranteed.*

2) Since the proposed method forces the closed loop characteristic polynomial to satisfy $\|\tilde{d}_{cl}\|_1<1$, it follows that the resulting controller internally stabilizes the loop. Note however that internal stability does not prevent the appearance of stable pole/zero cancellations. This leads to the following question: Suppose that an s–order controller has been found such that the closed–loop system achieves $(s+r)$–equalized performance μ^{s+r}. Assume that some zero pole cancellations occur so that the resulting closed loop has a minimal realization of order $n'<n=s+r$. Does this reduced plant achieve the same equalized n'–performance level? The answer is not necessarily. This should not be surprising, since the equalized performance framework does not assume zero initial condition. However, Theorem 5 guarantees that the reduced plant (of order $n'<n$) still achieves an n–equalized performance level less or equal than μ^{s+r}.

7 Conclusions

In this paper we propose a new approach to the problem of synthesizing fixed order controllers for optimally rejecting persistent disturbances. This approach is based upon the idea of *equalized performance*: A plant has an equalized performance level μ if whenever N consecutive output values are below μ, then the ℓ^∞ norm of the entire output sequence is guaranteed to be less or equal μ. This can be thought of as a extension of the usual ℓ^1 performance criterion, and indeed both coincide in the case of plants having a finite impulse response.

By exploiting a characterization of equalized performance in terms of the coefficients of an ARMA model of the plant, we have shown that the problem of synthesizing *fixed order* controllers that optimize performance (in the equalized sense) reduces to solving a Linear Programming problem whose size can be determined a–priori. An additional feature of our method is that it can be used even in cases where the plant has zeros on the stability boundary. On the other hand, it is well known that the traditional ℓ^1 methodology breaks down in these cases, leading to discontinuities in the cost [8].

An important open question is the extension of the method to the MIMO case. In principle this could be accomplished by means of a vector ARMA model. Clearly, the definitions in the paper could be easily rephrased in a vector sense by requiring that for any output string $e(0),e(1),\ldots,e(n-1)$ whose element norms are all below μ, the norm of $e(n)$ is also below μ. However, the extension looses the physical meaning of the SISO equalized performance in the following sense: the first order multivariable system

$$A=\left[\begin{array}{c} a \end{array}\right],\quad B=\left[\begin{array}{cc} 1 & 1 \end{array}\right],\quad C=\left[\begin{array}{c} 1\\ 1 \end{array}\right],\quad D=0,$$

could be associated to the equation

$$\left[\begin{array}{c} e_1(k+1)\\ e_2(k+1) \end{array}\right]=a\left[\begin{array}{c} e_1(k)\\ e_2(k) \end{array}\right]+\left[\begin{array}{c} w_1(k+1)\\ w_2(k+1) \end{array}\right].$$

However, it is immediately apparent that a true correspondence between this ARMA model and the original state space system does not exist, since in the former the output components are related by $e_1=e_2$. Thus the extension of the method to the MIMO case does not appear to be trivial.

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