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Worst case l^∞ to l^∞ gain minimization: dynamic versus static state feedback.

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Abstract

It has been recently shown that for the problem of optimal rejection of persistent disturbances using full-state feedback, static non-linear controllers can recover the performance level achieved by any linear dynamic controller. In this paper we complement these results by showing that for this problem nonlinear dynamic time-varying finite dimensional, possibly discontinuous, compensators do not offer any advantage over memoryless time invariant nonlinear-compensators. Moreover, we show that the best possible (over the set of all stabilizing controllers) disturbance rejection can be achieved by using globally Lipschitz piecewise-linear controllers.

1 Introduction

A common problem arising in many engineering applications is to design controllers capable of stabilizing a given linear time invariant system while, at the same time, minimizing the worst case response to some exogenous disturbances. When the signals involved are persistent bounded signals, with size measured in terms of peak time-domain values, it leads to l^1 optimal control theory ([1] [9] [10]).

The l^1 theory is appealing because it directly incorporates time-domain specifications. However, l^1 optimal controllers can have arbitrarily high order. This fact, along with the well known fact that for the full-state feedback case both $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control problems admit static (sub)optimal controllers, prompted the study of full-state feedback optimal l^1 controllers. To the best of our knowledge, this problem was first addressed in a set-theoretic framework in the early 1970's [3] [14]. However, this line of research was abandoned, due probably to the complexity of the resulting controller, which was not compatible with the computer technology available at that time.

More recently, it was shown in [12] that the optimal *linear* l^1 state-feedback controller can be dynamic, with arbitrarily high McMillan degree. However, if the class of admissible controllers is expanded to include non-linear control laws, then the performance achieved by any internally stabilizing dynamic *linear* state-feedback controller can be recovered using *static non-linear* state feedback [17]. Constructive procedures to synthesize this static non-linear feedback law have been presented in [2][3][4] [8] [18]. A question that arises then is whether or not nonlinear controllers can outperform linear controllers.

Denote by μ_{LTI}, μ_{NLS} and μ_{NLTV} the optimal l^∞ to l^∞ induced operator norms over the sets of causal norm-bounded linear time-invariant compensators, the set of non-linear static compensators and the set of non-linear time-varying compensators respectively. In [16] it was shown that $\mu_{LTI} = \mu_{NLTV}$ for the cases where either i) $T_2 = I$ (or has a stable inverse), or ii) the non-linear compensator is differentiable at the origin. If these conditions fail, then we can have $\mu_{NLTV} < \mu_{LTI}$, as shown by a simple static counterexample in [19] involving a system of the form:

$$z(t) = \bar{C}w(t) + \bar{D}u(t), \quad y(t) = w(t), \tag{1}$$

for which $\mu_{NLS} < \mu_{LTI}$. This example can be easily extended to the *state feedback case* by using delay augmentation to introduce dynamics as follows:

$$x(t+1) = w(t),\ z(t) = \bar{C}x(t)+\bar{D}u(t),\ y(t) = x(t). \tag{2}$$

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Since the augmentation sees no feedback, any feedback $u = Qy$ for (1) produces the same norm as the state feedback $u = Qx$ for (2). Thus we have that for this case $\mu_{NLS} < \mu_{LTI}$. This is due to the fact that in the state feedback case the condition $T_2 = I$ usually fails. While this example shows that in the state–feedback case non–linear static controllers can improve the performance of linear controllers, the question of whether or not arbitrary non–linear time–varying, non smooth compensators can improve the performance of static non–linear controllers is still open. The main result of this paper shows that for the state–feedback case this question has a negative answer, i.e. $\mu_{NLTV} = \mu_{NLS}$. It follows that when searching for the optimal controller, the search can be limited to *memoryless* non–linear controllers. Moreover, within this class, performance arbitrarily close to optimal can always be achieved by using a piecewise linear, globally Lipschitz controller. This result will be established by taking into account the more general case in which the system is affected also by parametric memoryless model uncertainty.

2 Existence of suboptimal static state feedback controllers

In the sequel we denote by $\|\cdot\|_p$, $p = 1, \infty$ the p norm for vectors and $\|\cdot\|_{l^p}$, the l^p norm for sequences. Given a set $S \subset R^n$ and $\nu \in R$, we denote by $\nu S \doteq \{\nu x, x \in S\}$ and by $\overline{conv}\{S\}$ the closure of its convex hull. Given a point $x \in R^n$, we define its distance to the set S as $Dist(x, S) = \inf_{y \in S} \|x - y\|$.

Definition 1 *Consider the discrete-time dynamic system*

$$x(t+1) = f(x(t), u(t), d(t)) \tag{3}$$

where $x(t) \in R^n$, $u \in \Sigma_u \subset R^m$, and $\|d(t)\|_{l^\infty} \leq 1$. A convex, bounded set P containing the origin in its interior is said to be $\lambda - \Sigma_u-$contractive for this system if for all $x \in P$ there exists $u \in \Sigma_u$ such that $f(x, u, d) \in \lambda P$, $0 \leq \lambda \leq 1$ for all $\|d\|_\infty \leq 1$. In the special case $\lambda = 1$, P is said to be Σ_u–invariant [13].

Remark 1 *In the special case of autonomous systems of the form*

$$x(t+1) = f(x(t), d(t)) \tag{4}$$

this definition reduces to the usual definition of λ–contractivity (positive invariance in the $\lambda = 1$ case), i.e., for all $x \in P$, $f(x, d) \in \lambda P$, for all $\|d\|_\infty \leq 1$. For simplicity and by a slight abuse of notation, we will use the term λ–contractive (positively invariant) for both systems (3) and (4) when the meaning is clear from the context.

Definition 2 *Consider the autonomous system (4). Given a sequence $d = \{d(0), d(1), \ldots\}$ and an initial condition x_o, denote by $\phi(t, x_o, d)$ the solution at the time t. The origin–reachable state set R_∞ is defined as $R_\infty \triangleq \{\xi : \xi = \phi(t, 0, d),\ t > 0,\ \|d\|_{l_\infty} \leq 1\}$.*

Consider the linear uncertain plant:

$$\begin{aligned} x(t+1) &= A(w(t))x(t) + B_1 d(t) + B_2(w(t))u(t) \\ z(t) &= Cx(t) + D_{11}d(t) + D_{12}u(t) \end{aligned} \tag{5}$$

where $x(t) \in R^n$, $u(t) \in R^q$, $w(t) \in R^s$, $d(t) \in R^m$ and $z(t) \in R^p$ represent the state of the system, the control input, parametric model uncertainty, the exogenous disturbances and the controlled outputs respectively. We assume that $A(w)$ and $B_2(w)$ are of the form:

$$A(w) = \sum_{i=1}^{s} A_i w_i, \quad B_2(w) = \sum_{i=1}^{s} B_{2_i} w_i, \quad w \in \mathcal{W}, \tag{6}$$

with

$$\mathcal{W} \doteq \{w : \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0\},$$

and that $w(t)$ is a memoryless parameter. Suppose that a stabilizing nonlinear full state (possibly time-varying) feedback compensator of the form:

$$\begin{aligned} \hat{x}(t+1) &= f(x(t), \hat{x}(t), t) \\ u(t) &= h(x(t), \hat{x}(t), t) \end{aligned} \tag{7}$$

is given. Note that we do not require any regularity assumption for this compensator which may even be discontinuous. It is well known that closing the loop with the controller (7) is equivalent to applying the state feedback:

$$\begin{bmatrix} u(t) \\ \hat{u}(t) \end{bmatrix} = \begin{bmatrix} h(x(t), \hat{x}(t), t) \\ f(x(t), \hat{x}(t), t) \end{bmatrix} \doteq \bar{\Phi}(x, \hat{x}, t). \tag{8}$$

to the following augmented linear system

$$\begin{aligned} \begin{bmatrix} x(t+1) \\ \hat{x}(t+1) \end{bmatrix} &= \begin{bmatrix} A(w(t)) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \\ &+ \begin{bmatrix} B_1 \\ 0 \end{bmatrix} d(t) + \begin{bmatrix} B_2(w(t)) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u(t) \\ \hat{u}(t) \end{bmatrix} \end{aligned} \tag{9}$$

$$z(t) = [C \quad 0] \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + D_{11} d(t) + [D_{12} \quad 0] \begin{bmatrix} u(t) \\ \hat{u}(t) \end{bmatrix}$$

Denote the closed–loop system by:

$$\begin{aligned} \xi(t+1) &= \bar{F}(\xi(t), \bar{\Phi}(x, \hat{x}, t), w(t), d(t)), \\ z(t) &= \bar{G}(\xi(t), \bar{\Phi}(x, \hat{x}, t), w(t), d(t)) \end{aligned} \tag{10}$$

where

$$\xi(t) \doteq \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

In the sequel, we will assume that the controller (7) is such that the resulting closed–loop system has the following properties:

Property (i): Finite l^∞ to l^∞ induced gain:

$$\sup_{\|d\|_{l^\infty}\leq 1, \xi(0)=0} \|z\|_{l^\infty} \doteq \mu < \infty \tag{11}$$

For convenience we add a "small" perturbation $\tilde{d}$, resulting in the following modified system:

$$\begin{aligned} \xi(t+1) &= \bar{F}(\xi(t), \bar{\Phi}(x, \hat{x}, t), w(t), d(t)) + \tilde{d}(t), \\ z(t) &= \bar{G}(\xi(t), \bar{\Phi}(x, \hat{x}, t), w(t), d(t)), \\ \|\tilde{d}(t)\|_\infty &\leq \delta, \end{aligned} \tag{12}$$

Denote by $\tilde{R}_\infty^{(\delta)}$ the 0-reachability set of (12). Note that $R_\infty \subset \tilde{R}_\infty^{(\delta)}$. We require the following additional two properties for the modified system.

Property (ii): Continuity of $\tilde{R}_\infty^{(\delta)}$ with respect to full dimension perturbations: For each $\epsilon_1 > 0$ there exists $\delta > 0$ such that

$$\sup_{\xi\in\tilde{R}_\infty^{(\delta)}} Dist(\xi, R_\infty) \leq \epsilon_1 \tag{13}$$

Property (iii): Continuity of the cost with respect to full dimension perturbations: For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\tilde{\mu} \doteq \sup_{\substack{\|d\|_{l^\infty}\leq 1 \\ \|\tilde{d}\|_{l^\infty}\leq\delta \\ \xi(0)=0}} \|z\|_{l^\infty} \leq \mu + \epsilon \tag{14}$$

Note that assumptions (ii) and (iii) are both reasonable and desirable from a practical standpoint. However, they can be weakened at the price of much more involved proofs.

Lemma 1 *For every $\epsilon > 0$ there exists $\delta > 0$ and a static control*

$$\begin{bmatrix} u \\ \hat{u} \end{bmatrix} = \begin{bmatrix} v(x,\hat{x}) \\ \hat{v}(x,\hat{x}) \end{bmatrix} \doteq \eta(\xi) \tag{15}$$

that renders the set

$$\tilde{S} = \overline{conv}\{\tilde{R}_\infty^{(\delta)} \bigcup -\tilde{R}_\infty^{(\delta)}\}$$

positively invariant for the closed-loop system (12) and guarantees the following output bound:

$$\|z\|_\infty = \|Cx + D_{11}d + D_{12}v(x,\hat{x})\|_\infty \leq \mu + \epsilon, \tag{16}$$

for all d such that $\|d\|_\infty \leq 1$.

Proof. First note that $\tilde{R}_\infty^{(\delta)}$ is an invariant set for the system (12). Given admissible sequences w, d and $\tilde{d}$, denote by $\phi(t, 0, w, d, \tilde{d})$ the corresponding trajectory originating at $\xi = 0$. By construction, given any point $\xi_1 \in \tilde{R}_\infty^{(\delta)}$ there exists a finite $t > 0$ and admissible sequences w, d, $\tilde{d}$, such that $\phi(t, 0, w, d, \tilde{d}) = \xi_1$ and $\phi(t+1, 0, w, d, \tilde{d}) \in \tilde{R}_\infty^{(\delta)}$ for all $w(t) \in \mathcal{W}$, $\|d(t)\|_\infty \leq 1$ and $\|\tilde{d}(t)\|_\infty \leq \delta$. It follows that the control law $\eta_1(\xi) \doteq \bar{\Phi}(x, \hat{x}, t)$ renders the set $\tilde{R}_\infty^{(\delta)}$ invariant for the system (9). Consider now a point $\xi = -\phi(t, 0, w, d, \tilde{d}) \in -\tilde{R}_\infty^{(\delta)}$. Clearly the control $\eta_2(\xi) \doteq -\eta_1(-\xi)$ renders this set invariant. Now, any point ξ in the set $\tilde{S}$ can be written as convex combination of elements of $\tilde{R}_\infty^{(\delta)}$ and $-\tilde{R}_\infty^{(\delta)}$:

$$\xi = \alpha\xi_1 + \beta\xi_2 \tag{17}$$

where the scalars $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, and where $\xi_1 \in \tilde{R}_\infty^{(\delta)}$ and $\xi_2 \in -\tilde{R}_\infty^{(\delta)}$. To this point we associate the control:

$$\eta(\xi) = \begin{bmatrix} v(\xi) \\ \hat{v}(\xi) \end{bmatrix} \doteq \alpha\eta_1(\xi_1) + \beta\eta_2(\xi_2) \tag{18}$$

Since the controls η_1 and η_2 map ξ_1 and ξ_2 in $\tilde{R}_\infty^{(\delta)}$ and $-\tilde{R}_\infty^{(\delta)}$ respectively, for all $w \in \mathcal{W}$, $\|d\|_\infty \leq 1$ and $\|\tilde{d}\|_\infty \leq \delta$, it follows from the linearity of (9) and the convexity of $\tilde{S}$, that the control η guarantees that $\xi' = \bar{F}(\xi, \eta(\xi), w, d) + \tilde{d} \in \tilde{S}$, for all $w \in \mathcal{W}$, $\|d\|_\infty \leq 1$ and $\|\tilde{d}\|_\infty \leq \delta$.
To establish the output bound, we recall that properties (i) and (iii) imply that for all $\xi \in \tilde{R}_\infty^{(\delta)}$ we have $\phi(t, 0, w, d, \tilde{d}) = \xi$ for some $t > 0$ and that $\|z(t)\|_\infty = \|\bar{G}(\xi, \bar{\Phi}(\xi, t), w(t), d(t))\|_\infty \leq \mu + \epsilon$. Taking the control $\eta(\xi)$ as above we have that $v(x, \hat{x}) = h(x(t), \hat{x}(t), t)$ and thus we get

$$\|z(t)\|_\infty = \|Cx + D_{11}d + D_{12}v(x,\hat{x})\|_\infty \leq \mu + \epsilon.$$

By symmetry for $-\xi \in -\tilde{R}_\infty^{(\delta)}$, the control $-\eta(\xi)$ guarantees the same output bound. Now for any $\xi \in \tilde{S}$ we take the control (18) and thus

$$\begin{aligned} \|z\|_\infty &= \|C[\alpha\xi_1 + \beta\xi_2] + D_{11}d \\ &+ D_{12}[\alpha v(x_1, \hat{x}_1) + \beta v(x_2, \hat{x}_2)]\|_\infty \\ &\leq \mu + \epsilon \end{aligned} \tag{19}$$

□

Since the set $\tilde{S}$ is invariant for the modified system (12), it is clearly invariant for the original system (10). However the positive invariance property alone is not sufficient to guarantee asymptotic stability of the resulting closed–loop system. In the following lemma we establish closed–loop asymptotic stability by showing that $\tilde{S}$ is a contractive set for the original closed-loop system (10).

Lemma 2 *The static function $\eta(\xi)$ defined in (15) is such that there exists $\lambda < 1$ such that for every $\xi \in \tilde{S}$,*

$$\bar{F}(\xi, \eta(\xi), w, d) \in \lambda\tilde{S} \tag{20}$$

for all $w \in \mathcal{W}$ and $\|d\|_\infty \leq 1$.

Proof. Define the δ-ball as $\mathcal{B}_\delta = \{\xi : \|\xi\|_\infty \leq \delta\}$ and define the set (see [17] [3] [5])

$$\tilde{S}' = \{\xi : \xi + \tilde{d} \in \tilde{S}, \quad \text{for all } \tilde{d} \in \mathcal{B}_\delta\}.$$

Since B_δ has full rank, $\tilde{S}'$ is in the interior of $\tilde{S}$. Since $\tilde{S}'$ and $\tilde{S}$ are both convex and compact sets and since $\tilde{S}$ contains the origin in its interior, it follows that there exists a positive $\lambda < 1$ such that $\tilde{S}' \subset \lambda\tilde{S}$. Consider first the perturbed system (12). The control $\eta(\xi)$ guarantees that, for all $w \in \mathcal{W}$, $||d||_\infty \le 1$ and $||\tilde{d}||_\infty \le \delta$, $\xi' = \bar{F}(\xi, \eta(\xi), w, d) + \tilde{d} \in \tilde{S}$. Hence, we have that for the original system (10), $\bar{F}(\xi, \eta(\xi), w, d) \in \tilde{S}' \subset \lambda\tilde{S}$. This means that $\tilde{S}$ is λ–contractive for system (10). □

Define now the projection $\tilde{P}$ of $\tilde{S}$ as:

$$\tilde{P} = \{x : \exists\, \hat{x} : \xi = [\ x^T \quad \hat{x}^T\ ]^T \in \tilde{S}\}. \qquad (21)$$

Clearly the set $\tilde{P}$ can be rendered contractive for the original system (5) by any selection Φ in the set-valued map $V(x)$ defined as:

$$\Phi(x) \in V(x) = \{v(x, \hat{x}), \quad \text{for some } \hat{x} \text{ s.t. } [\ x^T \quad \hat{x}^T\ ]^T \in \tilde{S}\} \qquad (22)$$

Since the control (15) guarantees that, for all $\xi \in \tilde{S}$, $\xi' = \bar{F}(\xi, \eta(\xi), w, d) \in \lambda\tilde{S}$, for all $w \in \mathcal{W}$ and $||d||_\infty \le 1$ and since the projection of $\lambda\tilde{S}$ is $\lambda\tilde{P}$, it follows from (9) that for all $w \in \mathcal{W}$ and $||d||_\infty \le 1$

$$x' = A(w)x + B_2(w)\Phi(x) + B_1 d \in \lambda\tilde{P}. \qquad (23)$$

Now note that since condition (16) does not depend explicitly on $\hat{x}$[1], it holds for all $x \in P$. Thus $\Phi(x)$ is such that the output bound (16) is satisfied, for all d such that $||d||_\infty \le 1$

$$||z||_\infty = ||Cx + D_{11}d + D_{12}\Phi(x)||_\infty \le \mu + \epsilon. \qquad (24)$$

We can now summarize the results of this section in the following theorem.

Theorem 1 *Assume that the system (5) with the dynamic time-varying control (7) satisfies conditions (i)–(iii). Then, for each $\epsilon > 0$ there exists a stabilizing (in the sense that if $d(t) = 0$ then $x(t) \to 0$) static nonlinear control $u = \Phi(x)$ such that the resulting closed–loop l^∞ to l^∞ induced gain does not exceed $\mu + \epsilon$.*

Proof. We have already established the existence of a static control law $u = \Phi(x)$ rendering the compact set $\tilde{P}$ contractive. Since $\tilde{P}$ contains the origin in its interior, it follows that for any trajectory such that $x(0) = 0$, the control law $u = \Phi(x)$ guarantees that $x(t) \in \tilde{P}$, for all $w(t) \in \mathcal{W}$ and $||d||_{l^\infty} \le 1$. Since the output bound (16) is satisfied for every $x \in \tilde{P}$, it follows that the l^∞ to l^∞ induced norm does not exceed $\mu + \epsilon$. The fact that the control $\Phi(x)$ asymptotically stabilizes the system can be established proceeding as in [5], by exploiting the fact that since the 0-symmetric set $\tilde{P}$ contains the origin as an interior point, it induces a norm that is a Lyapunov function for the system. □

[1] In the sense that it enters the expression only through v

To derive a Lipschitz control, we consider the following constructive procedure, originally introduced in [15]. Assume that a contractive polytope $\tilde{P}'$ containing the origin in its interior is known. For each vertex x of the polytope there exists a control vector u such that $A(w)x + B_2(w)u + B_1 d \in \lambda\tilde{P}'$ for all $||d||_\infty \le 1$. Partition now the state space into the conic sectors generated by the positive combinations $\mathcal{S} = \{x = \sum_{i=1}^n \alpha_k x_{i_k},\ \alpha_i \ge 0\}$ of a n-tuple of vertices $\{x_{i_k},\ k = 1, \dots, n\}$ belonging to the same facet of $\tilde{P}'$. These sectors can be selected such that $\bigcup_h \mathcal{S}_h = R^n$ and $\mathcal{S}_h \bigcap \mathcal{S}_k$ has empty interior if $k \ne h$. For a given sector $\mathcal{S}_h$, let X_h and U_h denote the matrices having as columns the coordinates of the vertices of the sector and the corresponding control action respectively, and consider the linear gain $K_h = U_h X_h^{-1}$. Then the variable structure controller $u = K_{h(x)}x$ renders the polytope $\tilde{P}'$ invariant [15]. Moreover, this control law is globally Lipschitz in R^n [6]. We use these results to show that for the full state feedback problem discontinuous controllers do not offer any advantages over continuous, globally Lipschitz ones.

Theorem 2 *Under the assumptions of Theorem 1, given any $\epsilon > 0$ there exists an internally stabilizing globally Lipschitz controller such that the resulting closed–loop system has an l^∞ to l^∞ induced norm not exceeding $\mu + \epsilon$.*

Proof. From Lemma 4.2 in [6] we have that the set $\tilde{P}$ can be always approximated by a contractive polyhedron $\tilde{P}'$ included in $\tilde{P}$, in the sense that for each $0 < \delta < 1$ there exists a contractive set $\tilde{P}'$ such that $(1-\delta)\tilde{P} \subset \tilde{P}' \subset \tilde{P}$. The proof follows now by considering the variable structure controller $u = U_h X_h^{-1} x$ associated with the vertices of $\tilde{P}'$ and proceeding as in Theorem 1 with the set $\tilde{P}$ replaced by $\tilde{P}'$. □

Since $\tilde{P}'$ is contractive, this control guarantees global convergence of the state to $\tilde{P}'$ (and if $d(t) = 0$ to the origin).

3 Construction of a static controller

We turn our attention now to the problem of synthesizing the control law. To this effect we introduce the additional assumption that the performance output is of the form

$$z(t) = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} D_{11}^1 \\ D_{11}^2 \end{bmatrix} d(t) + \begin{bmatrix} 0 \\ D_{12}^2 \end{bmatrix} u(t). \qquad (25)$$

From a practical standpoint, by including the control output among the performance variables, we rule–out the possibility of the controller requiring an unrealistically large control effort. Moreover, we will assume that C_1 has full column rank. Note that this assumption is not restrictive because if this is not the case, we can always add fictitious output variables, with "small" weights (see [17]), in order to complete the rank of C_1. Recall that achieving l^∞ to l^∞ induced gain not exceeding μ is equivalent to keeping state and control inside the polyhedral set [8]:

$$\Sigma^*(\mu) = \{x, u: \; \|Cx + D_{11}d + D_{12}u\|_\infty \leq \mu, \forall \|d\|_\infty \leq 1\}. \tag{26}$$

It can be shown (see [8] for details) that $\Sigma^*(\mu)$ is defined by the set of inequalities

$$\|C_i x + D_{12_i} u\|_\infty \leq \mu - \|D_{11_i}\|_1 = \theta_i, \quad i = 1, \ldots, p, \tag{27}$$

where C_i and D_{12_i} denote the i-th rows of C and D_{12}. Due to the structure of the output (25), the set Σ^* is the cartesian product of two polyhedral convex sets containing the origin in their interior, $\Sigma_u(\mu)$ and $\Sigma_x(\mu)$, each one obtained from the inequalities involving u and x respectively. Thus, given μ and $\lambda < 1$, the algorithm proposed in [5] can be used to generate S^λ_{max} (which can be possibly empty), the largest λ–$\Sigma_u(\mu)$—contractive set included in $\Sigma_x(\mu)$. Moreover, S^λ_{max} can be approximated arbitrarily close by a polyhedral set S' that can then be used to generate the control action, as outlined in the last section. Note that the algorithm in [5] requires the set $\Sigma_x(\mu)$ to be compact. This motivates the assumption on C_1. It can be easily removed by modifying the procedure to handle the case where $\Sigma_x(\mu)$ is unbounded.

Henceforth, denote by μ_{opt} the smallest μ guaranteeing the bound (i). From the previous section we have that there exists a contractive set $\tilde{S} \subset \Sigma_x(\mu_{opt} + \epsilon)$. Note that the existence of such a set is clearly a necessary and sufficient condition for the algorithm in [5] to produce a non-empty set. From the results of the previous section we have that:

a) if $\mu < \mu_{opt}$ the largest invariant (i.e. contractive with $\lambda = 1$) set in $\Sigma_x(\mu)$ with $u \in \Sigma_u(\mu)$ is empty;

b) if $\mu > \mu_{opt}$ there exists $\lambda < 1$ such that the largest λ-contractive set is not empty. Moreover, it can be shown that for all $\lambda' \geq \lambda$) the set is polyhedral.

With this in mind, we can synthesize the control law by setting first $\lambda = 1$ and constructing the largest invariant set and then augmenting or reducing μ depending on whether or not this set is empty. This leads to upper (μ^+) and lower (μ^-) bounds of μ_{opt} that can be arbitrarily refined. Then by a proper $\lambda < 1$ sufficiently close to 1, we assure stability.

4 Example

In this section we illustrate the features of the proposed controller with a simple example. Consider the uncertain system having the following state–space realization:

$$x(t+1) = \begin{bmatrix} 1+w & 1 \\ -1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} d(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} d(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

where the uncertain time varying parameter w satisfies $|w| \leq \nu$. Note that in this example C_1 does not have full column rank. Thus, as mentioned before, in order to apply the proposed synthesis procedure we need to add a small additional fictitious output. We selected as additional output $\tilde{z} = 10^{-3}\, x_2$, which does not affect the solution.

ν	μ_{opt}	n_p
0.0	6.000	8
0.1	6.719	12
0.2	7.789	12
0.3	9.539	12
0.4	12.53	8
0.5	18.99	8
0.6	94.78	8

Table 1: Optimal l^1 norm versus ν

Table 1 shows the optimal l^1 norm for different values of $\nu \leq \nu_{stab} = 0.615$, the maximal level for which the system can be stabilized, and n_p the number of planes characterizing the corresponding invariant region. From this table we have that for $\nu = 0.4$, the optimal closed–loop l^∞ to l^∞ gain is $\mu = 12.53$. For this value of μ, the vertices of the invariant region $P = \{x : x = X\alpha, \;\; \|\alpha\| \leq 1\}$ are given by the columns of the matrices X and $-X$, (ordered in such a way that $v_{2i} = -v_{2i-1}, \;\; i = 1 \ldots 4$), where

$$X = \begin{bmatrix} -7.8135 & -0.1476 & 0.0000 & 7.8135 \\ 4.1640 & -6.5683 & -6.6273 & -6.1064 \end{bmatrix}$$

Finally, Figure 1 shows the set S^λ_{max} with $\lambda = 0.999$ included in the set $\Sigma_x(12.53)$, and Table 2 shows the different gains that constitute the variable structure controller (note that by construction opposite sectors have associated the same gain)

5 Conclusions

In this paper we consider the problem of persistent disturbance rejection via full–state feedback. This problem has attracted considerable attention since it was

Sector	Gain
1-3/2-4	[-.0558 -1.754]
3-5/4-6	[-0.696 -1.739]
5-7/6-8	[0.116 -1.739]
1-8/2-7	[-0.400 -2.400]

Table 2: The sector gains

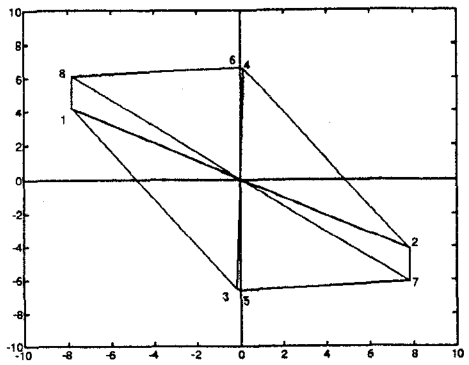

Figure 1: The set S^{λ}_{max}

shown in [12] that state–feedback optimal l^1 controllers can be dynamic, with arbitrarily high order. Recent work [7, 17, 18] has shown that the performance level achieved by any *finite–dimensional* linear dynamic controller can be recovered by a *nonlinear, static* controller, in other words $\mu_{NLS} \leq \mu_{LTI}$. While it is known that for certain problems $\mu_{NLS} = \mu_{LTI}$, this is not true in general for the state feedback case, as we have shown with a simple counterexample.

In this paper we complete these results by showing that the performance level achieved by any non–linear dynamic, possibly time varying full–state feedback controller can be also recovered via static state feedback. Moreover, within this class, the controller can be restricted to be piecewise linear.

While these results were derived for the discrete-time case, they can be extended to the continuous-time case proceeding along the same lines but the proofs are considerably more involved. Alternatively, from a constructive point of view, the properties of the Euler approximating system (EAS) [6, 7, 8] can be used to generate contractive sets for continuous-time systems, proceeding as in [6]. Thus, under appropriate assumptions, the results presented in this paper hold also in the continuous-time case.

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