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Persistent Disturbance Rejection via Static State Feedback

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Abstract

In contrast with $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control theories, the problem of persistent disturbance rejection (l^1 optimal control) leads to dynamic controllers, even when the states of the plant are available for feedback. Using viability theory, it has recently been shown, in a non-constructive way [17], that in the state-feedback case, the same performance achieved by *any* dynamic linear time invariant controller can be achieved using memoryless non-linear state feedback. In this paper we give an alternative, constructive proof of these results for discrete and continuous time systems. The main result of the paper shows that in both cases, the l^1 norm achieved by any stabilizing state-feedback linear dynamic controller can be also achieved using a memoryless variable structure controller.

1 Introduction

A large number of control problems involve designing a controller capable of stabilizing a given linear time invariant system while minimizing the worst case response to some exogenous disturbances. This problem is relevant for instance for disturbance rejection, tracking and robustness to model uncertainty (see [19] and references therein). When the exogenous disturbances are modeled as bounded energy signals and performance is measured in terms of the energy of the output, this problem leads to the well known $\mathcal{H}_\infty$ theory. On the other hand, if performance is measured in terms of the peak value of the output, it leads to $\mathcal{H}_2$ theory. Finally, the case where the signals involved are persistent bounded signals, with size measured in terms of the peak time–domain values, leads to the l^1 optimal control theory, formulated by Vidyasagar [19, 20], and solved by Dahleh and Pearson both in the discrete and continuous time cases [7, 8], by using duality to recast the problem into a linear–programming form.

The l^1 theory is appealing because it directly incorporates time–domain specifications. Moreover, it furnishes a complete solution to the robust performance problem. However, in contrast with $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control where it is well known that (sub)optimal controllers having the same order of the plant can be found and that a separation principle holds, l^1 optimal controllers can have arbitrarily high order. It has been shown through examples [10] that, even in the state feedback case, where (sub)optimal $\mathcal{H}_\infty$ and $\mathcal{H}_2$ *static* controllers can be found, optimal linear l^1 controllers can be dynamic and of arbitrarily high order.

Restricting the compensator to be linear does not entail any loss of performance, since it has been recently shown [16] that, in terms of the l^1 cost, nonlinear compensators offer no advantage over linear feedback. However, recent work by Shamma [17] shows that non–linear feedback can in [17] that the l^1 cost achieved by any *dynamic* linear full state feedback controller can be also achieved by using a *memoryless* non–linear state feedback.

The present paper is motivated by these results, which although furnishing an important existence result, are non–constructive. Here we show that, given any *dynamic* full state feedback controller achieving an l^1 cost μ_d, a memoryless variable structure controller can be found achieving the same cost, and we give an explicit expression for this controller, both for discrete and continuous time systems. Moreover, we establish some connections and give a perspective on some earlier results on disturbance rejection using state feedback controllers.

The problem of the rejection of persistent disturbances using state feedback has been considered as far back as 1970, [2, 1, 12]. These papers addressed the problem of finding a static state feedback control, possibly under control input constraint, guaranteeing the permanence of the state in an given time–dependent set, under set-constrained disturbances. The problem was solved by finding a sequence of sets (the reduced target tube) in which the state could be confined by means of an appropriate control action. This idea can be used to find optimal controllers, by computing the target tube for increasing values of the disturbance bounding set, until such a set is found to be empty. Unfortunately in general the target tube is not a polyhedron, even in the case in which the constraints sets are polyhedra (this is the main link with the l^1 theory since the unit ball of the infinity norm is a polyhedron). Although the target sets could be approximated by using ellipsoids, this approximation is usually rather conservative. Thus, the set of control actions that maintain the state confined to the ellipsoidal approximations could be empty, even though the problem is feasible. This theory was abandoned, probably in view of the computational complexity which was not apparently compatible with the computer technology of that time. Only recently, these results where reconsidered and extended to periodic systems [18], in connection with distribution systems.

The main result of this paper shows that a finitely determined polyhedral invariant set can be constructed based on the optimal linear l^1 controller. Projecting this set onto the state space we find a (polyhedral) set, which plays the role of the target tube. This set induces a piecewise linear controller (i.e. a controller which is linear in any simplicial sector of the set), which stabilizes the system and yields an l^1 cost which is ϵ away from the optimal cost, where ϵ can be made arbitrarily small. These results are extended to the continuous–time case, using the results in [6].

The paper is organized as follows. In Section 2 we introduce the notation to be used and we formally state the problem. In section 2.3 we show how to compute a

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finitely determined positively invariant set N for the closed loop system obtained using the optimal linear l^1 controller. Section 3 contains the main result of the paper. Bounded disturbance rejection is achieved by confining the states of the plant to the set N. In turn, this can be achieved by using a piecewise linear controller. In section 4 we extend these results to the continuous-time case. Section 5 presents a simple example. Finally, in section 6 we present some concluding remarks.

2 Preliminaries

2.1 Notation and Definitions

Given a matrix A, we denote by A_i its i-th row. For $x \in R^n$ we define $|x|$ as the vector with components $|x_i|$. We denote the 1−norm as $\|x\|_1 \triangleq \sum_{i=0}^{n} |x_i|$ and the infinity norm as $\|x\|_\infty \triangleq \max_i |x_i|$. l^1 denotes the space of absolutely summable sequences $h = \{h_i\}$ equipped with the norm $\|h\|_1 \triangleq \sum_{k=0}^{\infty} |h_i| < \infty$. l_∞ denotes the space of bounded sequences $h = \{h_i\}$ equipped with the norm $\|h\|_\infty \doteq \sup_{k \geq 0} |h_i| < \infty$. We denote by l_∞^p the space of bounded vector sequences $\{h(k) \in R^p\}$. In this space we define the norm $\|h\|_\infty \doteq \sup_i \|h_i(k)\|_\infty$. Assume now that $H : l_\infty^q \rightarrow l_\infty^p$ is a bounded linear operator defined by the usual convolution relation $y = H * u$. If we denote by $H(k)$ the Markov parameters of H, its induced $l_\infty^q \rightarrow l_\infty^p$ norm is given by:

$$\|H\|_1 \doteq \max_i \sum_{j=1}^{n} \|h_{ij}\|_1 \;=\; \max_i \sum_{k=o}^{\infty} \|h_i(k)\|_1$$

Assume now that the operator is proper rational and has a state–space realization (A, B, C, D). Then $\|(A, B, C, D)\|_1$ denotes $\|H\|_1$.

Definition 1 *Consider the discrete-time dynamic system $x(t+1) = f(x(t), d(t))$ where $x(t) \in R^n$ and where d is an element of the unit ball of l_∞. A convex, compact set P containing the origin is said to be $\lambda-$contractive for this system if for all $x \in P$ we have $f(x, d) \in \lambda P$, $0 \leq \lambda \leq 1$ for all $\|d\|_\infty \leq 1$. In the special case $\lambda = 1$, then P is said to be positively invariant.*

2.2 Statement of the Problem

Consider the linear time–invariant plant:

$$\begin{aligned} \mathcal{D}x(t) &= Ax(t) + B_1 d(t) + B_2 u(t) \\ z(t) &= Cx(t) + D_{11} d(t) + D_{12} u(t) \end{aligned} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^q$, $d(t) \in R^m$ and $z(t) \in R^p$ represent the state of the system, the control input, the exogenous disturbances and the controlled outputs respectively. As usual, the symbol $\mathcal{D}$ represents the operator $\mathcal{D}\{x(t)\} = \{x(t+1)\}$. Given an internally stabilizing linear dynamic state feedback controller with state–space realization:

$$\begin{aligned} \mathcal{D}w(t) &= A_K w(t) + B_K x(t) \\ u(t) &= C_K w(t) + D_K x(t) \end{aligned} \tag{2}$$

the corresponding closed–loop system is given by:

$$\begin{aligned} \mathcal{D}\xi(t) &= A_C \xi(t) + B_C d(t) \\ z(t) &= C_C \xi(t) + D_C d(t) \end{aligned} \tag{3}$$

where:

$$\begin{aligned} \xi &\doteq \begin{bmatrix} x \\ w \end{bmatrix} \\ A_C &\doteq \begin{bmatrix} A + B_2 D_K & B_2 C_K \\ B_k & A_K \end{bmatrix} \quad B_C \doteq \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\ C_C &\doteq \begin{bmatrix} C + D_{12} D_K & D_{12} C_K \end{bmatrix} \quad D_C \doteq D_{11} \end{aligned} \tag{4}$$

where A_C is a stable matrix. Assume that the controller is such that the closed loop l_∞ to l_∞ induced norm is equal to μ, i.e:

$$\sup_{\|d\|_\infty \leq 1} \|z\|_\infty = \mu \tag{5}$$

Then, the problem that we address in this paper is the following:

Problem 1 *Given $\epsilon > 0$, find an internally stabilizing static feedback control $u(t) = \Phi(x(t))$ such that the l_∞ to l_∞ induced norm of the closed loop system does not exceed $1 + \epsilon$.*

In the sequel we will show constructively that this problem admits a piecewise linear solution. In order to establish this result we will first introduce some preliminary results, giving a geometrical interpretation of (5).

2.3 Preliminary Results

Proposition 1 *Consider the set:*

$$\Xi(\mu) = \{\xi \colon |C_C \xi| \leq \mu \bar{1} - \delta\}, \tag{6}$$

where $\bar{1} \doteq [\ 1\ 1\ \ldots\ 1\ ]^T \in R^p$ and $\delta \in R^p$ is the vector whose i-th component is given by $\delta_i \doteq \|D_{C_i}\|_1$. Then equation (5) holds if and only if the origin-reachable state set R_∞ of the system with $\|d\|_\infty \leq 1$ is included in the set $\Xi(\mu)$.

Proof. First notice that (5) holds iff R_∞ is included in the set [17] $\left\{\xi \colon |C_C \xi + D_C d| \leq \eta \doteq \bar{1}, \forall\, d : \|d\|_\infty \leq 1\right\}$. But it is easily shown that this set coincides with the set $\Xi(\mu)$.

□

Remark 1 *The number μ is the smallest positive having the property that $R_\infty \subseteq \Xi(\mu)$.*

Proposition 2 *Let E be any matrix (possibly a zero matrix) such that the pair $\left(\begin{bmatrix} C_c \\ E \end{bmatrix}, A_c\right)$ is observable. Then the set R_∞ is included in the set*

$$N = \{\xi \colon |E\xi| \leq \nu, |C_c \xi| \leq \mu \bar{1} - \delta\}$$

where $\nu \doteq \|(A_C, B_C, E, 0)\|_1$.

Proof. Since the system (3) is asymptotically stable, R_∞ is a bounded set. The statement follows immediately from Proposition 1 by including the components of $z' = E\xi$ among the controlled outputs.

□

The set N introduced in Proposition 2 is a convex and compact polyhedral set. It can be represented in the form:

$$N = \left\{\xi : |\Phi\xi| \leq \Gamma\right\} \tag{7}$$

where:

$$\Phi \doteq \begin{bmatrix} E \\ C_C \end{bmatrix}, \text{ and } \Gamma \doteq \begin{bmatrix} \nu\bar{1} \\ \mu\bar{1} - \delta \end{bmatrix}$$

Remark 2 *The set N will be used to guarantee asymptotic stability and disturbance rejection in cases where the pair (A_C, C_C) is not completely observable. If (A_C, C_C) is an observable pair, then E can be taken to be zero. In this case it suffices to consider $\Phi = C_C$ and $\Gamma = \mu\bar{1} - \delta$.*

The set N defined above has the property of including the origin-reachable set R_∞ of the closed-loop system. The set R_∞ is of course an invariant set for the closed-loop system. In fact it is the minimal invariant set since any invariant set must include it. The existence result of [17] is established by showing that there exist a memoryless state-feedback controller that renders this set viable. However, the set R_∞ is difficult to compute. In this paper we will pursue a different approach that furnishes a constructive proof of the result in [17] and an explicit expression for the non-linear controller. This approach requires the following steps: i) finding the maximally invariant (in the sense of containing any other invariant sets) included in N, ii) obtaining a λ-contractive polyhedron by perturbing N; iii) projecting this polyhedron into the plant-state space. This yields a viable polyhedron and an associated piecewise-linear asymptotically stabilizing control law.

Proposition 3 *For each $\epsilon > 0$ there exists $\lambda : 0 \leq \lambda < 1$ and an integer $\bar{k}$ such that the set*

$$S(\epsilon, \lambda, \bar{k}) = \Big\{\xi \;\; : \;\; |\,\Phi(\tfrac{A_C}{\lambda})^k\xi\,| \leq \Gamma + \epsilon\bar{1} - \sum_{j=0}^{k-1}\Delta^{(j)},\; k = 0, 1, \ldots, \bar{k}\Big\} \tag{8}$$

where $\Delta^{(j)}$ is the vector whose $i-th$ component is $\|[\Phi(\frac{A_C}{\lambda})^j\frac{B_C}{\lambda}]_i\|_1$, is a λ-contractive, convex, compact polyhedron containing the origin in its interior and such that:

$$S(\epsilon, \lambda) \subseteq N(\epsilon) \doteq \{\xi : \; \Phi\xi \leq \Gamma + \epsilon\bar{1}\} \tag{9}$$

Proof. The set $S(\epsilon, \lambda)$ is contractive for the system (A_C, B_C) if and only if it is an invariant set for the system $(\frac{A_C}{\lambda}, \frac{B_C}{\lambda})$. First, we show that the set $S(\epsilon, \lambda, \infty)$, if it is not empty, is invariant for $(\frac{A_C}{\lambda}, \frac{B_C}{\lambda})$. Equivalently, we must show that for any $\xi \in S(\epsilon, \lambda, \infty)$:

$$\hat{\xi} \doteq \frac{A_C}{\lambda}\xi + \frac{B_C}{\lambda}d \in S(\epsilon, \lambda, \infty) \text{ for all d: } \|d\| \leq 1$$

By substituting $\hat{\xi}$ in the k-th inequality defining the set $S(\epsilon, \lambda, \infty)$ in (8) we have:

$$|\,\Phi(\frac{A_C}{\lambda})^{k+1}\xi + \frac{B_C}{\lambda}d| \leq \Gamma + \epsilon\bar{1} - \sum_{j=0}^{k-1}\Delta^{(j)}, \text{ for all } d: \; \|d\| \leq 1.$$

This condition holds since it is strictly equivalent to the (k+1)-th inequality for ξ and $\xi \in S(\epsilon, \lambda, \infty)$. Next we show that $S(\epsilon, \lambda, \infty)$ is the maximal invariant set contained in $N(\epsilon)$. To this effect, assume that there exists a set $\hat{S}, S(\epsilon, \lambda, \infty) \subset \hat{S} \subseteq N(\epsilon)$. Consider a point $\hat{\xi} \in \hat{S}$, $\hat{\xi} \notin S(\epsilon, \lambda, \infty)$. Hence $\hat{\xi}$ violates the inequalities (8) for some k. Using the same argument as before we have that $\xi(t+1) = \frac{A_C}{\lambda}\hat{\xi}(t) + \frac{B_C}{\lambda}d(t)$ violates the (k-1)-th for some $d(t) : \; \|d(t)\| \leq 1$. Proceeding by induction we have that $\xi(t+k)$ violates the constraint 0 and therefore is not in $N(\epsilon)$ against the assumption that $\hat{S} \subseteq N(\epsilon, \lambda, \infty)$. Therefore $S(\epsilon, \lambda)$ is the maximal invariant set contained in $N(\epsilon)$ for $(\frac{A_C}{\lambda}, \frac{B_C}{\lambda})$, or equivalently, the maximal λ-contractive set for (A_C, B_C). From Proposition 2, R_∞ is included in N. Then, for any positive ϵ, we can chose λ smaller but sufficiently close to 1, such that every eigenvalue of A_c has modulus strictly less that λ and such that the reachability set $R_\infty(\lambda)$ of the system $(\frac{A_C}{\lambda}, \frac{B_C}{\lambda})$ is included in $N(\epsilon)$. This implies that the set $S(\epsilon, \lambda)$ is not empty. Moreover, by construction, (A_C, Φ) is observable and therefore the set $S(\epsilon, \lambda)$ is compact. From proposition 1, we have that $R_\infty \subseteq S(\epsilon, \lambda, \infty) \subset N(\epsilon)$ is equivalent to having each component of the output strictly bounded by $\Gamma_i + \epsilon$, i.e:

$$\max_i\{\frac{1}{\Gamma_i + \epsilon}\sum_{j=0}^{\infty}\|[\Phi(\frac{A_C}{\lambda})^j\frac{B_C}{\lambda}]_i\|_1\} < 1.$$

Hence the vector $\Gamma_\infty \doteq \Gamma + \epsilon\bar{1} - \sum_{j=0}^{\infty}\Delta^{(j)}$ has positive components and so $S(\epsilon, \lambda)$ contains the origin in its interior. Moreover, since the the right hand side of the inequalities in (8) is bounded below by Γ_∞ and the matrix $(\frac{A_C}{\lambda})$ is Hurwitz, it follows from the compactness of $S(\epsilon, \lambda, \infty)$ that there exists $\bar{k}$ such that for $k > \bar{k}$ the inequalities in (8) become redundant.

□

Remark 3 *It can be shown, (see [9] for details) that the number $\bar{k}$ in the proposition above can be chosen as the minimum integer having the property that the set in (8) does not change if $\bar{k}$ is increased. Thus, computing this set can be reduced to a linear programming problem. Moreover, we show in the sequel that in the case of the optimal l^1 controller, $\bar{k}$ is bounded by n_c, the dimension of the closed-loop system.*

Proposition 4 *Assume that A_C has only zero eigenvalues. Then $\bar{k} = n_c$ where n_C is the dimension of A_C.*

Proof. Follows immediately from the fact that A_C is a nilpotent matrix. □

3 Main results

In this section we use the results of proposition 3 to obtain a memoryless state feedback controller. This result will be established by finding an appropriate viable set P and the corresponding control action. Since the set $S(\epsilon, \lambda, \bar{k})$ is contractive, for each of its vertices ξ_i we have that:

$$A_C\xi_i + B_Cd \in \lambda S(\epsilon, \lambda, \bar{k}), \;\; \forall\; \|d\|_\infty \leq 1, \tag{10}$$

Denote by $\mathcal{P}$ the projection operator defined as $\mathcal{P}\xi = x$ and consider the set $P = \mathcal{P}[S(\epsilon, \lambda, \bar{k})]$. P is a convex, compact polytope, containing the origin in its interior, with vertices given by by $x_i = \mathcal{P}\xi_j$, for some $\xi_j \in vert\{S(\epsilon, \lambda, \bar{k})\}$. In the sequel we show that there exists a static feedback controller rendering the set P λ-contractive, and such that the l_∞ to l_∞ induced norm of the closed-loop system is less or equal $1 + \epsilon$. To construct this controller we start by finding an appropriate control vector for each vertex of P.

Proposition 5 *For each vertex x_i of P the control vector*

$$u_i = D_k x_i + C_K w_i = [D_K \; C_K]\xi_i \qquad (11)$$

is such that

$$Ax_i + B_1 d + B_2 u_i \in \lambda P, \qquad \text{for all } d\text{: } \|d\|_\infty \leq 1 \qquad (12)$$

and

$$\|z\|_\infty = \|Cx_i + D_{11}d + D_{12}u_i\|_\infty \leq (\mu + \epsilon) \qquad (13)$$

Proof. First note that the projection of $\lambda S(\epsilon, \lambda, \bar{k})$ is given by λP, so from (10) we have:

$$\mathcal{P}(A_C \xi_i + B_C d) \in \lambda P, \quad \forall \; \|d\|_\infty \leq 1$$

Equation (12) follows immediately from (3) and (11). By construction, $\xi_i \in N(\epsilon)$ defined in (9). Since the inequalities defining the set $\Xi(\mu + \epsilon)$ in (6) are a subset of those defining defining $N(\epsilon)$, it follows that $\xi_i \in \Xi(\mu + \epsilon)$. From proposition 1, we have that for all $d : \; \|d\|_\infty \leq 1$.

$$z_i(d) \doteq \|Cx_i + D_{11}d + D_{12}u_i\|_\infty = \|C_C\xi_i + D_C d\|_\infty \leq (\mu + \epsilon)$$

□

Now we are able to prove the main result of the paper. For each vertex x_i of P, equation (11) provides a control action that drives the state to the set $\lambda S(\epsilon, \lambda, \bar{k})$. Following an approach similar to that in [14, 4], we will exploit this property to synthesize a piecewise linear control which makes P contractive for the closed-loop system, hence guaranteeing both asymptotic stability and satisfaction of (5). Consider the family of matrices $X(h)$ defined by:

$$X(h) = [x_{k_1}^{(h)} x_{k_2}^{(h)} \ldots x_{k_n}^{(h)}], \; x_{k_j}^{(h)} \in vert\{P\}, j = 1, \ldots, n \quad (14)$$

obtained by selecting n different vertices of P, and the simplex

$$S_h = \left\{ x = \sum_{j=1}^{n} x_{k_j}^{(h)} \alpha_j, \sum_{j=1}^{n} \alpha_j \leq 1, \alpha_j \geq 0 \right\}.$$

(notice that the origin is a vertex of S_h). The vertices forming the matrices $X^{(h)}$ can be selected in such a way that S_h has a non-empty interior, $S_h \bigcap S_k$ has empty interior if $k \neq h$, and $\bigcup_k S_k = P$. Each point $x \in S_h$ is uniquely determined by the vector of coefficients α_j given by: $\alpha^{(h)} = [X^{(h)}]^{-1}x$ ($X^{(h)}$ is invertible since $int\{S_h\}$ is non-empty). Finally, for each matrix $X^{(h)}$ consider the matrix $U^{(h)}$ formed by considering the controls $u_{k_j}^{(h)}$ associated to each vertex by means of Proposition 5 and define the following linear-variable structure control [14]:

$$u = \Phi(x) = K^{(h)}x, \text{ with } K^{(h)} = U^{(h)}[X^{(h)}]^{-1} \qquad (15)$$

where h is such that $x \in S_h$. It is easily seen that this control action renders the set P λ–contractive. Moreover, it can be shown that the control function $\Phi(x)$ is Lipschitz on P [5]. By using these observations we can state now the main result of the paper:

Theorem 1 *Assume that the system (1) with the linear state feedback dynamic controller (2) is internally stable and satisfies the condition (5). Then for each $\epsilon > 0$ the static state nonlinear controller (15) is stabilizing and such that*

$$\sup_{\|d\|_\infty \leq 1} \|z\|_\infty \leq \mu + \epsilon$$

Proof. The fact that the control (15) is stabilizing if $0 \leq \lambda < 1$ is proved in [4]. We prove now that the closed-loop induced norm is bounded above by $\mu + \epsilon$. Denote by $x(t, 0)$ the trajectory corresponding to the initial condition $x(t) = 0$. Since P is invariant, it follows that $x(t, 0) \in P$ for all t. Since $P = \bigcup_h S_h$, $x(t, 0) \in S_h$ for some h, hence $x(t, 0) = X^{(h)}\alpha^{(h)}$. The corresponding control action is given by: $u = U^{(h)}[X^{(h)}]^{-1}x = U^{(h)}\alpha^{(h)}$. From (13) we have that

$$\begin{aligned} \|z(t)\|_\infty &= \|Cx + D_{11}d + D_{12}u\|_\infty = \\ &= \|CX^{(h)}\alpha^{(h)} + D_{11}d + D_{12}U^{(h)}\alpha^{(h)}\|_\infty \\ &\leq \sum_{j=1}^{n} \alpha_j^{(h)} \|Cx_{k_j}^{(h)} + D_{11}d + D_{12}u_{k_j}^{(h)}\|_\infty \\ &\leq (\mu + \epsilon) \text{ for all } \|d\|_\infty \leq 1 \end{aligned} \qquad (16)$$

□

Remark 4 *We recall that if a n-dimensional polytope S has n_f faces, then it has no more that $\binom{n_f}{n}$ vertices. The projection P of the polytope on a n'-dimensional subspace has no more than $n'_f \doteq \binom{n_f}{n - n'}$ faces and no more than $\binom{n'_f}{n'}$ vertices. However, these upper bounds are in general overly conservative. Tighter bounds exist for some classes of polytopes (see [13] for details). Likewise, the number of simplicial sectors can be bounded by a function of the number of vertices and the number of k-faces. We omit this discussion here for brevity. However we emphasize the fact that although the number of sectors is an exponential function of the problem data, consistent experience shows that the complexity of the controller tends to be reasonably low. For instance, for the example given in [17], the set $\bar{S}$ derived there coincides with the set P derived here. In this case, since the set is affine to a diamond (i.e. to a set with 2n vertices), Theorem 1 results in a linear static controller $u = Kx$, with $K = [\; -2/3 \quad 0 \;]$.*

Remark 5 *Alternatively, the control law can be computed on–line, by solving an optimization problem parametrized in x and having u as the unknown, as suggested in [4]. This alternative is particularly efficient in cases where the control dimension is low. In particular, for single control input systems the problem reduces to finding an admissible point in the intersection of several intervals.*

4 The continuous-time case

The result of last section can be easily extended to the continuous-time case as follows. Consider a system of the form (1) and the controller (2) where $\mathcal{D}$ represents now the derivative. We recall that, although in the continuous-time case, the optimal $\mathcal{L}^1$ solution is in general non rational [8], [6] and [15] provide procedures for synthesizing sub-optimal rational controllers, yielding $\mathcal{L}^1$ cost arbitrarily close to the optimum. By using the results of [6], we will show that, given a rational controller yielding an $\mathcal{L}^1$ cost μ then, there exists a non–linear static compensator of the form (15) such that the $\mathcal{L}_\infty$ to $\mathcal{L}_\infty$ induced norm of the closed–loop system is bounded by $\epsilon + \mu$, with ϵ arbitrarily small. To establish this result, we make use of the Euler approximating system (EAS). The EAS of the closed loop system (3) is defined as:

$$\begin{aligned}\xi(t+1) &= [I+\tau A_C]\xi(t)+\tau B_C d(t)\\ z(t) &= C_C\xi(t)+D_C d(t)\end{aligned} \qquad (17)$$

where τ is a positive parameter. We remark that applying the EAS of the compensator (2) to the EAS of (1) will result in the closed-loop system (17).

Proposition 6 *Assume that the continuous-time closed-loop system (3), where $\mathcal{D}$ is the derivative operator, is such that $\sup_{\|d\|_{\mathcal{L}_\infty}\le 1} \|z\|_{\mathcal{L}_\infty} = \mu$. Then, for each ϵ there exists a positive τ such that the l_∞ to l_∞ induced norm of (17) is bounded above by $\mu + \frac{\epsilon}{2}$.*

Proof. See [6].

In view of the theorem above, the following result is a straightforward application of the results of the previous section.

Theorem 2 *Assume that the continuous-time closed-loop system (3), where $\mathcal{D}$ is the derivative operator, is such that $\sup_{\|d\|_{\mathcal{L}_\infty}\le 1} \|z\|_{\mathcal{L}_\infty} = \mu$. Then, for each ϵ there exists a static state feedback of the form (15) such that the $\mathcal{L}_\infty$ to $\mathcal{L}_\infty$ induced norm of the closed-loop system is bounded above by $\mu+\epsilon$.*

Proof. From Proposition (6), we can find a discrete-time system with l_∞ to l_∞ induced norm equal to $\mu + \frac{\epsilon}{2}$. Using the result of Section 3, we have that there exists a linear-variable structure controller that achieves a cost $\mu+\epsilon$. Now we just have to apply this compensator to the continuous time system. Using the result of [5], we have that a) this control (which is Lipshitz) is stabilizing; and b) the set P derived for the discrete-time system is also an invariant set for the continuous-time system. So, applying exactly the same argument of the proof of Theorem 1, we conclude that $\|z\|_\infty \le \mu+\epsilon$, for all $d: \|d\|_\infty \le 1$.

□

Remark 6 *In [21] we furnished a procedure to find a value of the parameter τ such that the resulting rational controller will yield an l^1 cost $\le \mu+\epsilon$. The present procedure can be combined with this technique to find the value of τ.*

5 A Simple Example

Consider the third order system of Example 2 in [10], with $k = 2$ and $\gamma = 0$. A state-space realization of the plant is given by:

$$\left(\begin{array}{ccc|cc} 2.7 & -23.5 & 4.6 & 1 & 1\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ \hline 1 & -2.5 & 2 & 0 & 0\end{array}\right)$$

The optimal l^1 linear state-feedback is dynamic and has the following state space realization.

$$\left(\begin{array}{cc|ccc} -0.5819 & 0.7802 & -4.2265 & -3.2299 & 0.7954\\ -0.7629 & -0.5946 & 2.2154 & -2.4388 & 0.3995\\ \hline -6.1439 & -1.3136 & -1.5235 & 0 & 0\end{array}\right)$$

With this compensator, the l^1 cost is 4.2059. Setting $\lambda = 0.99$ we have that the cost of the modified system $(A_C/\lambda, B_C/\lambda, C_C, D_C)$ is equal to 4.3434. The pair (A_C, C_C) is observable, so in this case there is no need to consider a matrix E. Since the closed-loop system has all its poles at zero, $A_C^5 = 0$ and only ten delimiting planes are necessary to define the set $S(\lambda,\epsilon,5) = \{\Phi\xi \le \Gamma\}$ where Φ is the observability matrix of the closed loop system and $\Gamma = [\ 4.3434\ \ 3.3333\ \ 1.9830\ \ 1.9829\ \ 1.9829\ ]^T$. Projecting this set into the plant state space we derive the origin-symmetric set P, whose vertices are

$$\begin{aligned}
v_1 &= [\ -3.186\ \ -4.557\ \ -6.274\ ] = -v_2\\
v_3 &= [\ -1.163\ \ -3.218\ \ -1.269\ ] = -v_4\\
v_5 &= [\ 1.163\ \ 3.218\ \ 5.612\ ] = -v_6\\
v_7 &= [\ 1.243\ \ -1.173\ \ 0.084\ ] = -v_8\\
v_9 &= [\ 3.186\ \ 4.557\ \ 1.931\ ] = -v_{10}\\
v_{11} &= [\ 1.243\ \ 1.173\ \ 4.259\ ] = -v_{12}\\
v_{13} &= [\ 3.186\ \ -1.257\ \ -2.149\ ] = -v_{14}\\
v_{15} &= [\ 3.186\ \ 1.257\ \ -2.194\ ] = -v_{16}
\end{aligned}$$

These vertices form 28 symmetric simplicial sectors each one characterized by a triple of vertices. In each of these sectors, we apply a linear gain. The sectors and their associated gains are shown in Table 1.

GAIN	K_1	K_2	K_3	*SECTORS*
a	−1.092	22.85	−4.600	[1, 3, 6] − [2, 4, 5]
b	−1.092	22.85	−4.600	[1, 3, 10] − [2, 4, 9]
c	−1.891	24.42	−5.335	[1, 6, 13] − [2, 5, 14]
d	−2.025	23.50	−4.600	[1, 10, 13] − [2, 9, 14]
e	−1.458	22.98	−4.600	[3, 6, 12] − [4, 5, 11]
f	−1.458	22.98	−4.600	[3, 7, 12] − [4, 8, 11]
g	−1.563	22.89	−4.279	[3, 7, 16] − [4, 8, 15]
h	−1.156	22.58	−3.865	[3, 10, 16] − [4, 9, 15]
i	−1.885	23.69	−4.921	[5, 11, 14] − [6, 12, 13]
j	−1.713	22.71	−4.600	[7, 11, 14] − [8, 12, 13]
k	−1.713	22.71	−4.600	[7, 11, 16] − [8, 12, 15]
l	−1.713	22.71	−4.600	[7, 12, 15] − [8, 11, 16]
m	−1.713	22.71	−4.600	[7, 14, 15] − [8, 13, 16]
n	−2.025	23.50	−4.600	[10, 13, 16] − [9, 14, 15]

Table 1. The 14 Simplicial Sectors and their Corresponding Gains

Notice that due to the symmetry of the sectors, in principle only half as many different gains as the number of sectors are required. The resulting controller guarantees an l_∞ to l_∞ induced norm of 4.3434 for zero initial conditions. Moreover, if the initial state is outside the set P, then the controller guarantees a speed convergence of the state to P (in the sense defined in [4]) equal to $\lambda = 0.99$. The reader may observe that it turns out that several of the gains are equal even in sectors that are not symmetric. This is due to the special structure of P. There are examples where the gains are different in non-symmetric sectors. However, consistent experience shows that in general the gains tend to be very close to each other. This raises the interesting possibility of reducing the complexity of the controller by combining sectors, averaging their gains. For the example considered here, the single gain $K = [-1.607\ \ 23.077\ \ -4.6]$, obtained by averaging the gains over all sectors, internally stabilizes the plant and yields an l^1 cost of 4.7486, roughly 10% higher than the cost of the non-linear controller.

6 Conclusion

In contrast to $\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimal control theories, where the order of the optimal controller is bounded by the order of the plant, l^1 optimal controllers can have arbitrarily high order, even when the states of the plant are available for feedback. It is well known that the use of non-linear feedback will not improve upon the *performance* of a LTI controller. However, recent results using concepts from Viability theory [17] show that in the state feedback case, the same performance level can be achieved using *memoryless non-linear feedback.*

In this paper we give an alternative, constructive proof of these results and we show that the same level of disturbance rejection achieved with a linear dynamic controller can be achieved using memoryless piecewise-linear (i.e. variable structure) controllers, both in the discrete and continuous time cases.

We also establish some connections with earlier work on disturbance rejection. Note in passing that the results of section 2.3 extend the result of [11] on constructing maximally invariant sets to the case where the system is subject to persistent disturbances.

The example of section 5 highlights an important feature of the proposed controllers. Although the number of switching planes tends to be high, consistent numerical experience shows that in most cases the gains change little between adjacent sectors. As pointed out there, this raises the interesting possibility of reducing the complexity of the controller by combining sectors averaging their gains, eventually leading to static linear controllers. Research is currently being pursued in this direction.

References

[1] Bertsekas, D.P. and Rhodes, I.B., "On the minmax reachability of target set and target tubes", *Automatica*, Vol. 7, pp. 233-247, 1971.

[2] Bertsekas, D.P. "Infinite-Time Reachability of State-Space Regions by Using Feedback Control", *IEEE Transactions on Automatic Control*, Vol. AC-17, pp. 604-613, 1972.

[3] F. Blanchini. "Constrained control for perturbed linear systems", Proceedings of the 29-th Conference on decision and control, Honolulu, Hawaii, Dec. 5-7, 1990; See also "Constrained control for systems with Unknown disturbances", in Control and Dynamic Systems, Vol. 51, Edited by C.T. Leondes, Academic Press, 1992.

[4] F. Blanchini "Ultimate boundedness control for discrete-time uncertain systems via set-induced Lyapunov functions", *IEEE Trans. Automat. Contr.*, 1994, to be published.

[5] F.Blanchini. "Nonquadratic Lyapunov functions for robust control", submitted.

[6] F. Blanchini and M. Sznaier, "Rational $\mathcal{L}^1$ Suboptimal Compensators for Continuous-Time Systems," *IEEE Trans. Automat. Contr.*, vol AC-39, pp. 1487-1492, July 1994.

[7] M. A. Dahleh and J. B. Pearson, "l^1-Optimal Feedback Controllers for MIMO Discrete-Time Systems," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 314-322, April 1987.

[8] M. A. Dahleh and J. B. Pearson, "$\mathcal{L}^1$-Optimal Compensators for Continuous-Time Systems," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 889-895, October 1987.

[9] M. A. Dahleh and M. H. Kammash, "Controller Design for Plants with Structured Uncertainty," *Automatica*, Vol 29, 1, pp. 37-56, January 1993.

[10] I.J. Diaz Bobillo and M.A, Dahleh, " State feedback l^1 optimal controllers can be dynamic", *System and Control Letters*, Vol. 19, pp. 87-93, 1992.

[11] E. Gilbert and K. Tan, "Linear systems with sate and Control Constraints: the theory and the applications of the maximal output admissible sets", *IEEE Transactions on Autom. Contr.*, Vol.36, n.9,1991.

[12] Glover, J. D. and Schweppe, F. C. (1971), "Control of linear dynamic systems with set constrained disturbances", *IEEE Transactions on Automatic Control* , Vol.AC-16, pp. 411-423.

[13] B. Grunbaum, "Convex Politopes", John Wiley & Sons., London, 1967.

[14] P.O. Gutman and C. Cwikel, "Admissible Sets and Feedback control for Discrete-time linear dynamical systems with bounded controls and state", *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 373-376, April 1986.

[15] Y. Ohta, H. Maeda and S. Kodama, "Rational Approximation of $\mathcal{L}_1$ Optimal Controllers for SISO Systems," *IEEE Trans. Automat. Contr.*, vol. AC-37, pp. 1683-1691, November 1992.

[16] J. S. Shamma and M.A. Dahleh, "Rejection of persistent disturbances: nonlinear controllers", *Syst. and Contr. Lett.* Vol. 18, pp. 245-252, 1992.

[17] J. S. Shamma, "Nonlinear state feedback for l^1 optimal control", *Syst. and Contr. Lett.* Vol. 21, pp. 265-270, 1993.

[18] Ukovich, W.and Blanchini, F. (1993), "A linear programming approach to the control of discrete-time periodic systems with Uncertain Inputs", *Journal of Optimization Theory and Applications*, Vol.78, n.3, pp. 523-539.

[19] M. Vidyasagar, "Optimal Rejection of Persistent Bounded Disturbances," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 527-535, June 1986.

[20] M. Vidyasagar, "Further Results on the Optimal Rejection of Persistent Bounded Disturbances", *IEEE Trans. Automat. Contr.*, vol. AC-36, pp. 642-652, June 1991.

[21] Z. Q. Wang, M. Sznaier and F. Blanchini, "Further Results on Rational Approximations of $\mathcal{L}^1$-Optimal Controllers", *IEEE Trans. Automat. Contr,*. to appear, May 1995.