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Brief Paper

Robust Performance with Fixed and Worst-case Signals for Uncertain Time-varying Systems*

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Key Words—Robust performance; dynamic systems; unmodeled dynamics; time-varying systems.

Abstract—In this paper we focus our attention on the determination of upper bounds of the l^∞ norm of the output of a linear discrete-time dynamic system driven by a step input, in the presence of both persistent unknown, but, l^∞ bounded disturbances and memoryless time-varying model uncertainty. For the same type of systems we also analyze the transient behavior of the step response in terms of its overshoot. The problem is solved in a constructive way by determining appropriate invariant sets contained in a given convex region. Finally, we show how to extend these results to continuous-time systems. © 1997 Elsevier Science Ltd.

1. *Introduction*

In most practical situations the mathematical model of a dynamic system must include some uncertainties and disturbances due to unmodeled dynamics and/or time-varying conditions. In this paper, we investigate the problem of robust performance (in the l^∞ sense) of dynamic systems subject to parametric time-varying memoryless uncertainties and in the presence of l^∞ bounded disturbances. The problem of interest is to determine a bound on the worst-case l^∞ norm of the output due to a step input, and with zero initial conditions. Additionally, motivated by the case where no uncertainties are present, we are also interested in establishing whether or not the system exhibits an overshoot with respect to its steady state output value.

Khammash and Pearson (1991) provided robust performance conditions with respect to unknown but bounded disturbances. However, in many real problems, some design specifications are given in terms of the output to a given, fixed test signal (such as a step). Since the unit step belongs to the unit ball of l^∞, this problem can be addressed using the techniques proposed by Khammash and Pearson (1991) for structured dynamic uncertainty. However, this approach will yield a conservative bound, since these results provide the worst-case l^∞ bound of the output over the set of all possible l^∞ bounded inputs and dynamics.

The problem of robust step response performance under structured dynamic uncertainty has been addressed by Kammash (1994), where necessary and sufficient conditions for robust steady-state tracking have been provided, and by Elia *et al.* (1995), where separate lower and upper bounds for the maximum overshoot due to a given, fixed reference signal are given. These bounds are not tight in the sense of having a non-zero gap. This gap can be eliminated by assuming non-causal (i.e. not physically realizable) uncertainty blocks. Thus, applying those results to our problem where the uncertainty is restricted to be memoryless entails a double level of conservatism: the first due to the fact that the results by Elia *et al.* (1995) allow for dynamic uncertainty, and the second due to the fact that the conditions provided there are tight only for non-causal blocks.

In this paper, we provide a non-conservative bound for the case where the input signal is a step. This bound is obtained using a method based upon the construction of a suitable polyhedral region. These regions have been previously used in the context of robust stability analysis and synthesis (see, for instance, Barabanov, 1988; Blanchini, 1994, 1995; Blanchini and Miani, 1996; Bertsekas and Rhodes, 1971; Brayton and Tong, 1979, 1980; Michel *et al.*, 1984; Ohta *et al.*, 1993; Olas, 1991; Sznaier, 1993; Zelentsowsky, 1994 and references therein). Additionally, by exploiting this construction we present necessary and sufficient conditions for the existence of overshoot, and a way to compute both the steady-state output value and the overshoot in cases where the latter is present. The paper is organized as follows. In Section 2 we introduce some basic facts. In Section 3 we show how to obtain non-conservative bounds of the worst-case value of the step response in the presence of both bounded noise and parametric uncertainties. In Section 4 we exploit a similar technique to establish whether or not there exist overshoot, and if so, to compute it. Section 5 illustrates these results with a simple example. Finally, Section 6 contains some concluding remarks.

2. *Preliminaries*

2.1. *Notation.* Given a closed, convex set S we denote its interior as int$\{S\}$. A polyhedral set S will be represented either by a set of linear inequalities $S = \{x\colon F_i x \le g_i, i = 1, \ldots, s\}$, or by the dual representation in terms of its vertex set $\{x_i\}$, denoted by vert$\{S\}$. In the sequel we will use matrix notation to describe componentwise assignments as well as componentwise inequalities. Thus, in this notation a polyhedral set is expressed by the matrix inequality $S = \{x\colon Fx \le g\}$ where F is an $s \times n$ full column rank matrix and g represents an s-column vector. Finally, we denote by $\|\cdot\|$ the Euclidean norm in R^n while dist(x, S) denotes the distance of a point x from a set S, defined as dist$(x, S) = \inf_{y \in S} \|x - y\|$.

2.2. *Problem statement.* Consider the uncertain n-dimensional discrete-time system with m command inputs $u(k)$, q disturbance inputs $d(k)$ and p outputs:

$$\begin{aligned} x(k+1) &= A(w(k))x(k) + Bu(k) + Ed(k), \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

where $w(k)$ is an uncertain time-varying parameter, $A(w)$ is a matrix polytope of the form

$$\begin{aligned} A(w) &= \sum_{i=1}^{r} A_i w_i(k), \\ w(k) \in W &= \left\{ w\colon\ w_i \ge 0,\ \sum_{i=1}^{r} w_i = 1 \right\} \end{aligned} \tag{2}$$

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where A_i, B, E are given real matrices of appropriate dimensions and where the disturbance $d(k)$ belongs to D, the l^∞ unit ball, i.e. $D = \{d: \|d\|_{l^\infty} \leq 1\}$.

For these systems we are interested in determining a non-conservative bound for the peak value of the step response, i.e. the problem we address is the following:

Problem 1. Given the system (1) with zero initial state, and a step input of the form $u(k) = U$, $k \geq 0$, find

$$\mu_{\inf} = \inf\{\mu: \|y(k)\|_{l^\infty} \leq \mu \text{ for all sequences } w(k) \in W, \quad d(k),\ \|d\|_{l^\infty} \leq 1\}.$$

In order to simplify the exposition in the sequel we make the following assumption.

Assumption 1. There exists a matrix A_0 belonging to the matrix polytope such that the triplet (A_0, B, C) is reachable and observable.

Under Assumption 1 it is easily shown that a necessary and sufficient condition for Problem 1 to have a finite solution $\mu < +\infty$ is that the autonomous system $x(k+1) = A(w(k))x(k)$ is asymptotically stable. Thus, in the sequel we will limit our attention to asymptotically stable systems.

Definition 2.1. Given a convex compact set D, consider the system $x(k+1) = A(w(k))x(k) + Ed(k)$, where $d(k) \in D$. A set P in state space is positively D-invariant for this system if for every initial condition $x(0) \in P$ we have that $x(k) \in P$ for every $k \geqslant 0$, for every admissible disturbance $d(k) \in D$ and every admissible sequence $w(k)$.

Remark 2.1. Note that positive D-invariance is a property that depends both on the system and the admissible disturbance set D. In the sequel, for brevity we may sometimes omit an explicit reference to the system and talk about D-invariant sets whenever the system in question is clear from the context.

3. *Main results*

In this section we introduce the notion of *limit set*, i.e. the set to which all the trajectories "converge". We show that although from a theoretical point of view the performance of a system can be characterized by this set, this may not be practical due to the difficulty in computing it.

3.1. *Limit set.* Let us now introduce an "extended disturbances" system which treats the command inputs of system (1) as disturbances:

$$\begin{aligned} x(k+1) &= A(w(k))x(k) + E^{\bullet}d^{\bullet}(k), \\ y(k) &= Cx(k), \end{aligned} \tag{3}$$

where $E^{\bullet} = [B\ E]$, $d^{\bullet}(k) = [u^{\mathrm{T}}(k)\ d^{\mathrm{T}}(k)]^{\mathrm{T}}$ and the extended disturbance $d^{\bullet}(k)$ is constrained to belong to the polyhedral set

$$D^{\bullet} = \{[u^{\mathrm{T}}(k)\ d^{\mathrm{T}}(k)]^{\mathrm{T}}: u(k) = U,\ \|d(k)\|_{l^\infty} \leq 1\}.$$

We formalize now the definition of limit set that we use in the sequel.

Definition 3.1. Given the system (3), we will define the (possibly empty) limit set $\mathscr{L}$ as the set of all states x for which there exist admissible sequences w, $d^{\bullet}$ and a non-decreasing time sequence t_k such that

$$\lim_{k \to +\infty} \phi(0, t_k, w(\cdot), d^{\bullet}(\cdot)) = x,$$

where $\lim_{k\to+\infty} t_k = +\infty$ and $\phi(0, t_k, w(\cdot), d^{\bullet}(\cdot))$ denotes the value at the instant t_k of the solution of (3) originating at $x_0 = 0$ and corresponding to w and $d^{\bullet}$.

Lemma 3.1. If system (1) is asymptotically stable then the limit set $\mathscr{L}$ is non-empty and the state evolution of system [equation (3)], for every initial condition $x(0)$ and admissible sequences $d^{\bullet}(k) \in D^{\bullet}$ and $w(k) \in W$, converges to $\mathscr{L}$ (i.e. $\lim_{t\to\infty}\mathrm{dist}(x(t), \mathscr{L}) = 0$). Moreover, $\mathscr{L}$ is bounded and it is $D^{\bullet}$-invariant for system (3).

Proof. Boundedness and convergence follow immediately from the asymptotic stability of $A(w)$ which is equivalent to the existence of a norm $\|\cdot\|_*$ that is a Lyapunov function (Molchanov and Pyatnitskii, 1986;Blanchini and Miani, 1996) and such that the corresponding induced matrix norm satisfies $\|A(w)\|_* \leq \lambda < 1$. Denoting by $\Phi(k,h) = A(k-1)A(k-2)\ldots A(h)$ we have that $\|\Phi(k,h)\|_* \leq \lambda^{k-h}$. Thus,

$$\begin{aligned} \|x(k)\|_* &= \left\| \Phi(k,0)x(0) + \sum_{i=0}^{k-1} \Phi(k, i+1)E^{\bullet}d^{\bullet}(i) \right\|_* \\ &\leq \lambda^k \|x(0)\|_* + (1 + \lambda + \cdots + \lambda^{k-1}) \sup_{d^{\bullet} \in D^{\bullet}} \{\|E^{\bullet}d^{\bullet}\|_*\} \\ &\to \frac{1}{1-\lambda} \sup_{d^{\bullet} \in D^{\bullet}} \{\|E^{\bullet}d^{\bullet}\|_*\}. \end{aligned}$$

To establish invariance we need to show that for every $x \in \mathscr{L}$ we have

$$A(w)x + E^{\bullet}d^{\bullet} \in \mathscr{L}$$

for every $w \in W$ and $d^{\bullet} \in D^{\bullet}$. Suppose by contradiction that there exists $x \in \mathscr{L}$ such that $y = A(\bar{w})x + E^{\bullet}\bar{d}^{\bullet} \notin \mathscr{L}$ for some $\bar{w}$ and $\bar{d}^{\bullet}$. We will show that in this case for any arbitrary x_0 and $k > 0$ there exists $t > 0$ and appropriate sequences w and $d^{\bullet}$ such that the solution of equation (3) corresponding to the initial condition x_0 satisfies $\|x(t) - y\| \leq 1/k$. Setting $t_k = t$ and repeating the same argument taking as initial condition $x(t_k)$ we have that there exists $t_{k+1} > t_k$ such that $\|x(t_{k+1}) - y\| \leq 1/(k+1)$. Proceeding along this line we will have that $y \in \mathscr{L}$, leading to a contradiction.

For any arbitrary initial condition x_0, the corresponding state evolution is $x(t) = x_F(t) + x_L(t)$ where x_F and x_L denote the forced and free motions, respectively. From asymptotic stability we have that $\|x_L(t)\| \to 0$. Since from the definition of $\mathscr{L}\|x_F(t) - x\|$ can become arbitrarily small for appropriate sequences w, $d^{\bullet}$ and $t > 0$, the same property holds for $\|x(t) - x\|$. Let M denote the largest value over $w \in W$ of the induced matrix norm $\|A(w)\|$. Since $x \in \mathscr{L}$, by definition there exists $t > 0$ such that $\|x(t-1) - x\| \leq 1/kM$. Let $x(t) = A(\bar{w})x(t-1) + E^{\bullet}\bar{d}^{\bullet}$. Then

$$\begin{aligned} \|x(t) - y\| &= \|A(\bar{w})[x(t-1) - x]\| \\ &\leq \|A(\bar{w})\| \cdot \|(x(t-1) - x)\| \leq \frac{1}{k}. \end{aligned}$$

Finally, to establish closedness consider a sequence $y_k \in \mathscr{L}$ such that $\|y_k - y\| \leq 1/2k$. We need to show that $y \in \mathscr{L}$. Using the same argument as before we have that, since $y_k \in \mathscr{L}$, then for all x_0 and $k > 0$, there exist $t > 0$, and appropriate sequences w and $d^{\bullet}$ such that $\|x(t) - y_k\| \leq 1/2k$. Thus, the condition $\|x(t) - y\| \leq 1/k$ can be always achieved for arbitrary $k > 0$ and a sufficiently large t. □

Define now the set

$$X_0(\mu) = \{x: \|Cx\|_\infty \leq \mu\}. \tag{4}$$

A value $\mu < +\infty$ is said to be admissible if $\mu > \mu_{\inf}$. Clearly, a *necessary* condition for μ to be admissible is that $\mathscr{L} \subseteq X_0(\mu)$. This condition is not sufficient because even if it holds there may be trajectories starting from the origin outgoing from $X_0(\mu)$ and ultimately entering in it again to reach $\mathscr{L}$. Thus, knowledge of $\mathscr{L}$ alone does not give enough information to assess the complete (rather than asymptotic) system behavior. To compute the maximum overshoot one should reconstruct all possible trajectories starting from the origin, by propagating forward in time the effect of the uncertainties as shown by Barmish and Shankaran (1979), to reconstruct the reachability sets R_k (the set of all states that can be reached in k steps from the origin for all admissible w and d). However, as indicated by Barmish and Shankaran (1979), this technique leads to non-convex sets R_k. This difficulty can be circumvented by considering the sequence of convex-hulls $\bar{R}_k = \mathrm{conv}\{R_k\}$ rather than the sequence $\{R_k\}$. It can be shown that this sequence can be generated recursively and that it "converges" to $\bar{\mathscr{L}}$, the convex hull of $\mathscr{L}$. It is immediate to verify that the output of the system is bounded by μ if and only if $\tilde{R}_k \subseteq X_0(\mu)$, for all $k > 0$. However, proceeding in this way may not be realistic due to the large computational effort required to compute the sets $\tilde{R}_k$ and the lack of a reasonable stopping criterion (i.e. how many elements of the sequence $\tilde{R}_k$ to compute).

Thus, to solve the problem we will pursue a different approach leading to conditions related to a *single* convex set rather than a sequence. We state now the basic result of this section which will be used to give a solution to Problem 1.

Lemma 3.2. Given $\mu > 0$, the response of the system [equation (1)] to the input $u(k) = U$ satisfies $\|y\|_{l^\infty} \le \mu$ for every pair of sequences $w(k) \in W$, $d(k) \in D$ if and only if system (3) admits a D^*-invariant set P such that $0 \in P \subseteq X_0(\mu)$.

Proof. To show necessity consider the set P obtained by taking the convex hull of all the states belonging to the trajectories of (1) emanating from the origin for all possible sequences w and d when the system is driven by the step input $u = U$. The set P is contained in $X_0(\mu)$, contains the origin and it can be easily proven to be invariant for (3). The sufficiency is obvious. □

3.2. *Computation of the maximal D^*-invariant set.* Lemma 3.2 provides a general condition, given in terms of the existence of a D^*-invariant set P, $0 \in P \subseteq X_0(\mu)$, guaranteeing that a given performance level μ is achieved. In this subsection we provide a procedure to compute such a set if it exists. This is accomplished by finding the *maximal* D^*-invariant set for system (3), i.e. a set that contains any other invariant set in $X_0(\mu)$. The procedure relies on two stopping criteria (Theorems 3.1 and 3.2) that also allow to decide whether or not $0 \in P$.

Given a compact set S, we can define its preimage $C(S)$ as the set of all the states x that are mapped into S by the linear transformation $A(w)x + E^*d^*$, for all admissible $d^* \in D^*$ and w. If S is polyhedral with a matrix representation of the form $S = \{x: Fx \le g\}$, then $C(S)$ can be represented by

$$C(S) = \{x:\ F(A(w)x + E^*d^*) \le g,\ \text{for all } d^* \in D^* \text{ and } w \text{ satisfying equation (2)}\}.$$

Since the set D^* is itself polyhedral, the set $C(S)$ is defined by the following inequalities (Blanchini, 1994):

$$C(S) = \{x:\ FA_i x \le g - \delta,\ \ i = 1, \dots, r\},$$

where the components of the vector δ are given by

$$\delta_j = \max_{d^* \in D^*} F_j E^* d^*.$$

By recursively defining the sets $P^{(k)}$, $k = 0, 1, \dots$ as

$$P^{(0)} = X_0(\mu),\quad P^{(k)} = C(P^{(k-1)}) \cap P^{(k-1)},$$

it can be shown (Blanchini, 1994) that $P^{(\infty)}$ is the maximal D^*-invariant set contained in $X_0(\mu)$. We now introduce a theorem guaranteeing that this set can be expressed by a finite set of linear inequalities (i.e. it is polyhedral) and thus can be finitely determined.

Theorem 3.1. Suppose that system (3) is asymptotically stable. Then, if $\mathscr{L} \subset \operatorname{int}\{X_0(\mu)\}$ for some $\mu > 0$, the maximal D^*-invariant set contained in $X_0(\mu)$ is polyhedral. Moreover, in this case there exists k^* such that $P^{(\infty)} = P^{(k^*)}$ and this k^* can be selected as the smallest integer such that $P^{(k)}$ satisfies the vertex condition

$$A(w_h)x_j + E^*d_i^* \in P^{(k)}, \tag{5}$$

for every $x_j \in \operatorname{vert}\{P^{(k)}\}$, $d_i^* \in \operatorname{vert}\{D^*\}$ and $w_h \in \operatorname{vert}\{W\}$.

Proof. Consider the system $x(t+1) = A_0 x(t) + BU$ and let P_0^∞ denote its largest invariant set contained in $X_0(\mu)$. Assumption 1 implies that this set is compact (Tan and Gilbert, 1991). Let P^∞ denote the maximal D^*-invariant set of system (3) contained in $X_0(\mu)$. Since $P^\infty \subseteq P_0^{(\infty)}$ it follows that P^∞ is also compact. Moreover, $P^{(\infty)}$ coincides with the maximal invariant set contained in any compact polyhedral set S such that $P_0^{(\infty)} \subseteq S \subseteq X_0(\mu)$. From the stability of equation (3) we have that its state trajectories converge to $\mathscr{L} \subseteq \operatorname{int}\{X_0(\mu)\}$. Thus, proceeding as in Blanchini and Miani (1996), Lemma 3.1, it can be shown that there exists k^* such that $P^{(k^*+1)} = P^{(k^*)} = P^{(\infty)}$. Finally, the proof of equation (5) can be found in Blanchini (1992, 1994). □

Problem 1 can now be solved by determining the maximal D^*-invariant set contained in $X_0(\mu)$ for several values of μ and checking whether or not this set contains the origin. Then

- If $\mu > \mu_{\inf}$ we get a positive answer.
- If $\mu < \mu_{\inf}$ we get a negative answer.

Note that in both cases we get an answer in a finite number of steps, although there is no *a-priori* bound for such a number. In the first case this is due to Theorem 3.1. In the second case, this follows by the fact that the sequence of closed sets $P^{(k)}$ is ordered by inclusion and $P^{(\infty)}$ is their intersection. Thus, $0 \notin P^{(\infty)}$ if and only if $0 \notin P^{(k)}$ for some k.

Thus, the solution to Problem 1 can be obtained by starting from the initial set $X_0(\mu)$ and computing the sequence of sets $P^{(k)}$ until some appropriate stopping criterion is met. Note that the first positive criterion cannot be checked in a finite number of steps by propagating *forward* in time the reachability sets R_k, because at each instant k we cannot guarantee that the prescribed output level will not be violated in the future. The next theorem provides a new negative condition that will become fundamental in the next section, to address the overshoot problem.

Theorem 3.2. If the set $P^{(k)} \subset \operatorname{int}\{X_0(\mu)\}$ for some k, then the system (3) does not admit a D^*-invariant set contained in $X_0(\mu)$.

Proof. Suppose that there exists k such that $P^{(k)} \subset \operatorname{int}\{X_0(\mu)\}$ and system (3) admits an invariant region, and hence a maximal one $P^{(\infty)} \subset \operatorname{int}\{X_0(\mu)\}$. Define v as

$$v \doteq \inf_{x \notin X_0(\mu)} \operatorname{dist}(x, P^{(\infty)}).$$

For every initial condition $x_0 \notin P^{(\infty)}$ there exist sequences $\bar{w}$ and $\bar{d}^*$ such that the corresponding trajectory escapes from $X_0(\mu)$, i.e. $x(\bar{k}) \notin X_0(\mu)$ for some $\bar{k}$. Let $x(k)$ and $y(k)$ denote two system trajectories, corresponding to the same sequences $\bar{w}$ and $\bar{d}^*$ but different initial conditions. The updating equation for the difference $e(k) = x(k) - y(k)$ is

$$e(k+1) = A(\bar{w}(k))e(k). \tag{6}$$

Since (6) is stable, for any arbitrary $v > \varepsilon > 0$ there exists $\delta > 0$ such that, for $\|x(0) - y(0)\| < \delta$, we have $\|x(k) - y(k)\| \le \varepsilon$, for $k > 0$. On the other hand, we can take $x(0) \notin P^{(\infty)}$ and $y(0) \in P^{(\infty)}$ such that $\|x(0) - y(0)\| < \delta$. Now we have $y(\bar{k}) \in P^{(\infty)}$ and $x(\bar{k}) \notin X_0(\mu)$ which implies that $\|e(\bar{k})\| \ge v$ leading to a contradiction. □

These results suggest the following constructive procedure for finding a robust performance bound:

Procedure 3.1. The problem data are the system matrices, the input amplitudes U, the disturbance set D and a candidate output bound μ

0. Set $k = 0$ and set $P^{(0)} = X_0(\mu) = \{x:\ F^{(0)}x \le g^{(0)}\}$.
1. Consider the set $Q^{(k)} = \{x: F^{(k)}A_i x \le g^{(k)} - \delta^{(k)},\ i = 1, \dots, r\}$, where the vector $\delta^{(k)}$ has components $\delta_i^{(k)} = \max_{d \in D^*} F_h^{(k)} E^* d$, where $F_h^{(k)}$ is the hth row of $F^{(k)}$.
2. Compute the set $P^{(k+1)} = Q^{(k)} \cap P^{(k)}$. Let $P^{(k)} = \{x:\ F^{(k)}x \le g^{(k)}\}$.

3. If $0 \notin P^{(k+1)}$ or $P^{(k+1)} \subset \text{int}\{X_0(\mu)\}$ then stop, the procedure has failed. Thus, the output does not robustly meet the performance level μ.
4. If $P^{(k+1)}$ satisfies the vertex condition (5) stop (this implies $P^{(k+1)} = P^{(\infty)}$, the maximal D^*-invariant set).
5. Set $k = k + 1$ and go to step 1.

This procedure can then be used together with a bisection method on μ to approximate arbitrarily close the optimal value μ_{inf}, that solves Problem 1. In fact, if the procedure stops at step 3 we conclude that $\mu < \mu_{\text{inf}}$ and we can increase the value of the output bound μ. Else, if the procedure stops at step 4, we have determined an admissible bound for the output, say $\mu > \mu_{\text{inf}}$, that can be decreased. The procedure may fail to converge for the value $\mu = \mu_{\text{inf}}$. However, it can be shown that for any value of $\mu \neq \mu_{\text{inf}}$ the procedure terminates in a finite number of steps. Nevertheless, the possibility of an endless loop can be averted by putting an *a priori* limit on the number of iterations.

Remark 3.1. So far we have considered an output of the form $y(k) = Cx(k)$. These results can be easily extended to the proper plant case (i.e. $y(k) = Cx(k) + Pu(k) + Qd(k)$) by considering as initial set the polyhedron

$$\bar{X}_0(\mu) = \{x: |C_i x + P_i U| \le \mu - \|Q_i\|_1,\ i = 1, \cdot, p,\},$$

where $C_i P_i$ and Q_i denote the ith rows of C, P and Q respectively, and $\|\cdot\|_1$ denotes the 1-norm for vectors. As before, the peak value of the output is the smallest value of μ such that the largest invariant set in this region contains the origin.

4. *The overshoot and steady-state problem*

We have seen that arbitrarily good approximations of the l^∞ norm of the output of system (1), when driven by a step input of the form $u(k) = U$, can be obtained by checking whether or not system (3) admits a maximal invariant set contained in a suitable region. However, the l^∞ norm of the output does not provide a complete characterization of the system's performance during its transient. A better performance assessment can be accomplished by establishing whether or not the output exhibits overshoot and, in this case, by determining its value. If no uncertainties are present, the overshoot is measured with respect to the steady-state value. Since in our case the system under consideration is subject to uncertainty and affected by exogenous disturbances in addition to the reference input, before proceeding any further we must specify the definition of "steady-state output value" with respect to which the overshoot will be measured.

Definition 4.1. Consider system (1) driven by an input step of the form $u(k) = U$. The (upper) steady-state value of the system evolution is defined as

$$\mu_{ss} = \sup_{d^*, w} \limsup_{t \to \infty} \|Cx(t)\|_\infty. \tag{7}$$

The definition of overshoot for the systems under consideration in this paper is:

Definition 4.2. System (1) has (upper) overshoot if there exist sequences w and d such that $\mu_{\text{inf}} > \mu_{ss}$. In this case the positive number $\mu_{os} \doteq \mu_{\text{inf}} - \mu_{ss}$ is called the overshoot value. If $\mu_{os} = 0$ we say that the system has no overshoot.

The quantity defined above represents the difference between the worst-case peak and worst-case state values. Later, we will also briefly discuss an alternative definition of overshoot (the lower one) which is the difference of the worst-case peak and the lower steady-state value. Note in passing that in this context overshoot is related to a worst-case scenario. Thus, the system may exhibit overshoot only for some sequences w and not necessarily for all. With these definitions we are now able to introduce the steady-state and the overshoot/no-overshoot determination problem.

Problem 2. Establish whether or not system (1) exhibits overshoot, and if so determine μ_{os}.

The solution of Problem 2 is given by the following theorems and the corollary.

Theorem 4.1. The system (1) has a steady-state value $\mu_{ss} \le \mu$ if and only if the largest D^*-invariant region contained in $X_0(\mu)$ for system (3) is not empty.

Proof. If the maximal invariant set is not empty, since all the trajectories converge to $\mathscr{L}$ (Lemma 3.1), then $\mathscr{L} \subset X_0(\mu)$, thus sufficiency is obvious. For the necessity note that, by definition, the limit set $\mathscr{L}$ is such that for any $\bar{x} \in \mathscr{L}$, any large $\bar{t} > 0$ and any small $\varepsilon > 0$, there exist $t \geqslant \bar{t}$, w and d^* such that $\|x(t) - \bar{x}\| \le \varepsilon$. Suppose that the largest invariant set in $X_0(\mu)$ is empty, then necessarily $\mathscr{L} \nsubseteq X_0(\mu)$. Thus, take $\bar{x} \in \mathscr{L}$ such that $\bar{x} \notin X_0(\mu)$. Since $X_0(\mu)$ is a closed set then admissible sequences w and d^* exist such that the corresponding solution $x(t)$, starting from the origin, is outside $X_0(\mu)$ for arbitrary large values of t. It follows that $\mu_{ss} > \mu$. □

Theorem 4.2. The system (1) driven by an input of the form $u(k) = U$ has no overshoot if and only if for every $\mu \ge \mu_{ss}$ the maximal invariant set contained in $X_0(\mu)$ contains the origin.

Proof. Since $\mu \ge \mu_{ss}$ then system (3) admits a maximal D^*-invariant set $P \subseteq X_0(\mu)$. Suppose now that system (1) has no overshoot and that $0 \notin P$. This implies that the zero initial state evolution is such that $\|y\|_{l^\infty} > \mu$ for some sequence w and d^* (because otherwise $0 \in P$). The proof of the necessity follows now by noting that $\mu_{\text{inf}} > \|y\|_{l^\infty} > \mu \ge \mu_{ss}$, i.e. the system has overshoot. The sufficiency is obvious. □

Corollary 4.1. System (1) has overshoot if and only if there exists a value μ such that the maximal D^*-invariant set contained in $X_0(\mu)$ for system (3) is not empty and does not contain the origin. In this case, the overshoot value μ_{os} is the difference between the infimum of the values of μ for which the maximal invariant region in $X_0(\mu)$ contains the origin and the infimum value of μ for which the system admits a non-empty invariant region in $X_0(\mu)$.

These results can be combined with those of the previous section and the bisection method mentioned after Procedure 3.1 to obtain bounds on the l^∞ norms of both the overall trajectory and its asymptotic value. Note that, in addition to the critical cases where $\mu = \mu_{ss}$ and $\mu = \mu_{\text{inf}}$, we have the following three situations:

(i) $\mu < \mu_{ss}$: In this case $P^{(\infty)}$ is empty and this can be established in a finite number of steps because $P^{(\infty)}$ is the intersection of the closed sets $P^{(k)}$, ordered by inclusion. Thus, $P^{(\infty)}$ is empty if and only if emptiness of $P^{(k)}$ occurs for some finite k.
(ii) $\mu_{ss} < \mu < \mu_{\text{inf}}$: In this case the condition $0 \notin P^{(k)}$ occurs in a finite number of steps. Moreover, from Theorem 3.1 it follows that $P^{(\infty)} = P^{(k)}$ for some finite k.
(iii) $\mu_{\text{inf}} < \mu$: Again from Theorem 3.1 we have that this inequality can be established in a finite number of steps.

So far we have defined overshoot in terms of the upper steady-state value. Alternatively, the following definition can be considered. Assume, for simplicity, that the output y is scalar, and, without restrictions that the worst-case peak is approached by positive values of $y(k)$ (i.e. for all $\varepsilon > 0$ there exists k such that $\mu_{\text{inf}} - \varepsilon \le y(k) \le \mu_{\text{inf}}$). The (lower) steady-state value of the system evolution is defined as

$$\mu_{ss}^* = \inf_{d^*, w} \liminf_{t \to \infty} Cx(t). \tag{8}$$

The corresponding definition of lower overshoot is $\mu_{os}^* = \mu_{\text{inf}} - \mu_{ss}^*$. The following result characterizes this quantity.

Theorem 4.3. Define the set $\tilde{X}_0(\mu) \doteq \{x: \mu \le Cx \le \mu_{\text{inf}}\}$. Then $\mu_{ss}^* = \tilde{\mu}$, where $\tilde{\mu}$ is given by

$$\tilde{\mu} \doteq \sup\{\mu: \mathscr{L} \subset \tilde{X}_0(\mu)\}. \tag{9}$$

(note that this is equivalent to saying that $\tilde{\mu}$ is the largest value of μ such that the maximal invariant set in $\tilde{X}_0(\mu)$ is not empty). Moreover, if $\mathscr{L}$ has a non-empty interior (for instance, when Assumption 1 holds), the lower overshoot $\mu^*_{os} = \mu_{inf} - \mu^*_{ss}$ is positive.

Proof. Since $\mathscr{L}$ is closed, there exists x_0 such that $x_0 = \text{argmin}_{x \in \mathscr{L}} Cx = \tilde{\mu}$. From the definition of the limit set the quantity $\|x(k) - x_0\|$ can become arbitrary small for arbitrarily large k and for appropriate sequences w and d. This implies that for any $\varepsilon > 0$, $Cx(k) - Cx_0 \leq \varepsilon$, or equivalently $Cx(k) \leq \tilde{\mu} + \varepsilon$. Thus, we have $\mu^*_{ss} \leq \tilde{\mu} - \varepsilon$, which implies, since ε is arbitrary, that $\mu^*_{ss} \leq \tilde{\mu}$. The fact that $\mu^*_{ss} \geqslant \tilde{\mu}$ can be established in a similar way by invoking the convergence of the trajectories to $\mathscr{L}$. □

Remark 4.1. As a final remark we notice that the use of invariant regions allows us to extend these results to continuous-time systems $\dot{x}(t) = A(w(t))x(t) + Bu(t) + Ed(t)$ by introducing the Euler approximating system (EAS):

$$x(k+1) = [I + \tau A(w)]x(k) + \tau Bu(k) + \tau Ed(k), \quad \tau > 0. \quad (10)$$

It can be shown that as $\tau \to 0$ the maximal D^*-invariant set for this system converges to the maximal D^*-invariant set for the continuous-time system. This enables us to solve the continuous-time case by reducing it to an equivalent discrete-time problem for the EAS. Moreover, the values μ^{cont}_{ss} and μ^{cont}_{inf} for the continuous-time case are upper bounded by the values μ^{EAS}_{ss} and μ^{EAS}_{inf} computed for the EAS.

5. *Example*

To illustrate our results consider the following second-order system:

$$x(k+1) = \begin{bmatrix} 0.1 & 0.7 \\ -0.7 & 0.95 + \Delta(k) \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 0.3 \\ 0 \end{bmatrix} d(k),$$

$$y(k) = [0 \quad 3]x(k) \quad (11)$$

when $u(k)$ is the unity step, $\|d(k)\|_{l^\infty} \leq 1$ and $|\Delta(k)| \leq 0.4$.

Using Procedure 3.1 we computed the maximal invariant region contained in $X_0(\mu)$ for different values of μ and we found that the lower value for which the maximal invariant region contains the origin is $\mu^+_{inf} = 32.57$ (the tolerance used in the bisection method is $\varepsilon = 0.01$). Moreover, for $\mu = \mu^-_{inf} = 32.56$ there is $\bar{k}$ such that $P^{(\bar{k})}$ is contained in int$\{X_0(\mu)\}$. This implies (Theorem 4.2) that the system does not exhibit overshoot. Figure 1 shows the maximal invariant region of system (3) contained in $X_0(\mu^+)$. Surprisingly, the uncertain system (11) does not exhibit overshoot although both of the systems obtained by "freezing" the dynamics at either $\Delta = -0.4$ or $\Delta = 0.4$ have overshoot. In fact, we found the values $\mu^1_{inf} = 5.283$, $\mu^1_{ss} = 4.911$ for the first system and $\mu^2_{inf} = 22.206$, $\mu^2_{ss} = 20.492$ for the second. The maximal invariant regions for the two systems contained in $X_0(\mu)$ when μ takes the values μ^i_{ss} and μ^i_{inf}, $i = 1, 2$, are shown in Figs 2 and 3, respectively.

Our results can be compared against those by Elia *et al.* (1995) (where the uncertainties are assumed to be dynamic operators having l^∞ to l^∞ induced norm no greater than one) by eliminating the exogenous disturbances in the example above, and recasting it into the form shown in Fig. 4. Here the transfer matrix of the system is

$$\begin{bmatrix} \xi \\ z \end{bmatrix} = \frac{P}{1-P} \begin{bmatrix} W_1 & 1 \\ W_1 & 1 \end{bmatrix} \begin{bmatrix} \omega \\ r \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \omega \\ r \end{bmatrix},$$

where

$$P(z) = \frac{3z - 0.3}{z^2 + 1.95z + 0.285}$$

W_1 is a constant, and where $\omega = \Delta\xi$.

From Elia *et al.* (1995) bounds for the step response can be obtained by considering the spectral radius of appropriate matrices, whose entries can be computed starting from the block transfer functions M_{ij} and the reference signal r. Table 1 shows the values of $\gamma_{3.1}$ [a lower bound of the peak of the infinity norm of the step response for a causal perturbation Δ obtained from

Table 1. Comparison of different performance bounds

W_1	$\gamma_{3.1}$	$\gamma_{3.2}$	$\gamma_{4.4}$	γ_{gain}
0.133	∞	∞	∞	22.04
0.087	650.65	909.09	650.65	11.52
0.080	57.61	80.94	57.61	10.89
0.050	11.74	16.49	11.74	9.073

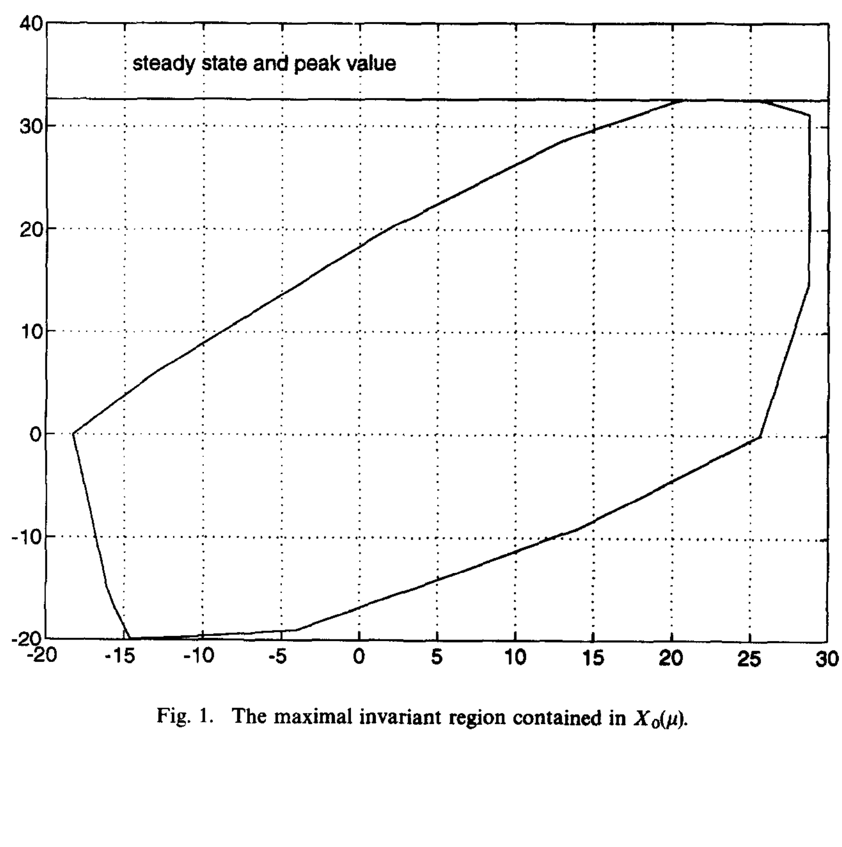

Fig. 1. The maximal invariant region contained in $X_0(\mu)$.

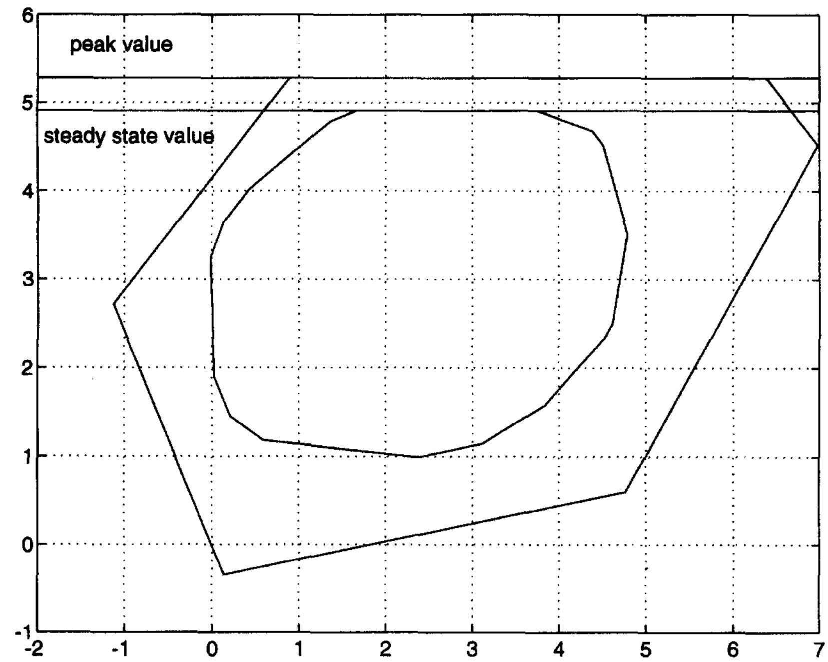

Fig. 2. The maximal invariant regions contained in $X_0(\mu_{ss}^1)$ and $X_0(\mu_{inf}^1)$ for system 1.

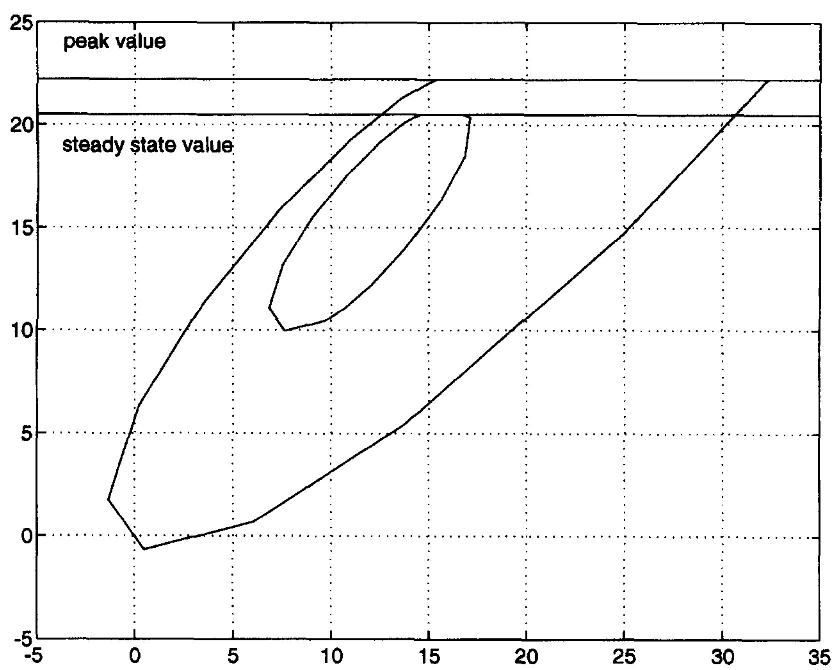

Fig. 3. The maximal invariant regions contained in $X_0(\mu_{ss}^2)$ and $X_0(\mu_{inf}^2)$ for system 2.

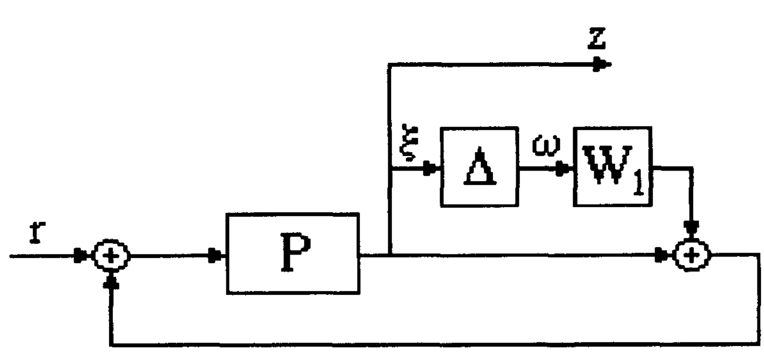

Fig. 4. The uncertainty setup.

Theorem 3.1 by Elia *et al.* (1995)]; $\gamma_{3.2}$ [the upper bound of the peak of the infinity norm of the step response for a causal perturbation Δ obtained from Theorem 3.2 by Elia *et al.* (1995)]; $\gamma_{4.4}$ [the worst-case peak of the infinity norm of the step response for a non-causal perturbation Δ obtained using Theorem 4.4 in Elia *et al.* (1995)]; and γ_{gain} (the worst-case peak of the infinity norm of the step response for time-varying memoryless perturbation Δ obtained using our results), for different values of the uncertainty amplitude W_1. As a final remark we stress once again that the large difference observed in these values is due to the conservatism entailed in recasting a problem involving memoryless time-varying gains into a dynamic l^∞ form.

6. *Conclusions*

This paper addresses the problem of robust performance (in the l^∞ sense) of dynamic systems subject to parametric time-varying uncertainties in the presence of l^∞ bounded disturbances. The main result of this paper provides a non-conservative robust performance bound for this case. This bound, obtained using a method based upon the construction of a suitable polyhedral region, can be computed in a finite number of steps. The drawback of the method is the potentially large number of constraints necessary to describe this region and the fact that the number of steps required to find it cannot be bounded *a priori*. On the other hand, as illustrated by the simple example, the performance bounds presented here are substantially less conservative than those achievable using previously proposed methods.

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