Rational $\mathcal{L}^1$ Suboptimal Compensators for Continuous-Time Systems

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Abstract—**The persistent disturbance rejection problem ($\mathcal{L}^1$ optimal control) for continuous time-systems leads to nonrational compensators, even for single input/single output systems [1]–[3]. As noted in [2], the difficulty of physically implementing these controllers suggest that the most significant application of the continuous time $\mathcal{L}^1$ theory is to furnish achievable performance bounds for rational controllers. In this paper we use the theory of positively invariant sets to provide a design procedure, based upon the use of the discrete Euler approximating system, for suboptimal rational $\mathcal{L}^1$ controllers with a guaranteed cost. The main results of the paper show that i) the $\mathcal{L}^1$ norm of a continuous-time system is bounded above by the l^1 norm of an auxiliary discrete-time system obtained by using the transformation $z = 1 + \tau s$ and ii) the proposed rational compensators yield $\mathcal{L}^1$ cost arbitrarily close to the optimum, even in cases where the design procedure proposed in [2] fails due to the existence of plant zeros on the stability boundary.**

I. Introduction

A large number of control problems involve designing a controller capable of stabilizing a given linear time-invariant system while minimizing the worst case response to some exogenous disturbances. This problem is relevant, for instance, for disturbance rejection, tracking, and robustness to model uncertainty (see [2] and references therein). When the exogenous disturbances are modeled as bounded energy signals and performance is measured in terms of the energy of the output, this problem leads to the well known $\mathcal{H}_\infty$ theory. The case where the signals involved are persistent bounded signals leads to the $\mathcal{L}^1$ optimal control theory, formulated and further explored by Vidyasagar [1], [3] and solved by Dahleh and Pearson both in the discrete [4] and continuous-time [2] cases.

The $\mathcal{L}^1$ theory is appealing because it directly incorporates time-domain specifications. Moreover, it furnishes a complete solution to the robust performance problem [5]. In contrast with the discrete time l^1 theory, however, the solution to the continuous-time $\mathcal{L}^1$ optimal control problem leads to nonrational compensators, even for single input/single output (SISO) systems. As noted by [2], the difficulty of physically implementing these controllers suggest that the most significant application of the continuous time $\mathcal{L}^1$ theory is to provide performance bounds for the plant.

Since it is well known that for discrete-time SISO systems l^1 theory leads to finite-dimensional controllers, an approach to synthesizing finite dimensional controllers for continuous-time systems is to use a discrete-time controller connected to the plant through sample and hold devices. Indeed, in the past few years, considerable attention has been focused on sampled-data systems ([6–9] and references therein). In particular, synthesis of suboptimal $\mathcal{L}^1$ controllers for these systems was addressed in [7], [8]. By using fast sampling, the sampled-data system is approximated with a time-varying discrete-time system, which is then "lifted" to yield a higher-dimensional, linear time-invariant (LTI) system. Finally, the suboptimal controller is designed by applying optimal l^1 theory to the lifted system. Hence, this approach results in finite-dimensional controllers (albeit time varying). It does not yield a guaranteed cost, however, in the sense that, due to intersampling effects, the l^1 norm of the discrete-time system used in the design process can be significantly lower than the $\mathcal{L}^1$ norm of the resulting closed-loop sampled-data system. Additionally, the use of sample and hold elements usually entail a performance loss, which may be significant, since the control is constrained to remain constant during the sampling period. While these effects can be alleviated by using fast sampling, this will result in very large order controllers (see [7] for details).

In this paper we use the theory of positively invariant sets to provide a design procedure, based upon the use of the discrete Euler Approximating System (EAS), for suboptimal rational $\mathcal{L}^1$ controllers. The main results of the paper show that i) the $\mathcal{L}^1$ norm of a continuous-time system is bounded above by the l^1 norm of the corresponding EAS, and ii) the optimal $\mathcal{L}^1$ system can be approximated arbitrarily close by a rational compensator related to the optimal l^1 compensator for the EAS by the simple transformation $z = 1 + \tau s$.

Our approach represents a significant departure from the sampled-data approach since it directly yields a continuous-time, LTI rational controller. Moreover, it provides a "guaranteed" cost, since the $\mathcal{L}^1$ norm of the resulting closed-loop system is bounded above by the l^1 norm of the auxiliary discrete-time system used to carry out the design.

The paper is organized as follows: In Section II we introduce the notation to be used, and we restate the main results concerning the $\mathcal{L}^1$ problem. In Section III we introduce the discrete-time EAS, and we propose a method for designing suboptimal rational controllers, yielding cost arbitrarily close to the optimal $\mathcal{L}^1$ cost, based upon the use of the optimal l^1 theory for the EAS. In Section IV we present a simple design example, and we compare our controller to the optimal $\mathcal{L}^1$ controller. Finally, while this paper was being reviewed, a completely different approach to synthesizing suboptimal rational $\mathcal{L}^1$ controllers was proposed by Ohta *et al.*, [10]. We explore the connections with our approach in Section V, where we also summarize our results and point to some open questions.

II. Problem Formulation and Preliminary Results

A. Notation

By $\mathcal{L}_\infty$ we denote the Lebesgue space of complex valued transfer matrices which are essentially bounded on the imaginary axis with norm $\|T(z)\|_\infty \triangleq \sup\{\sigma_{\max}[T(jw)]\}$. $\mathcal{H}_\infty$ denotes the set of stable complex matrices $G(s) \in \mathcal{L}_\infty$, i.e., analytic in $\Re(s) \geq 0$. $\mathcal{RH}_\infty$ denotes the subset of $\mathcal{H}_\infty$ formed by real rational transfer matrices. l_∞ denotes the space of bounded real sequences $\{e_k\}$ equipped with the norm $\|e\|_\infty \triangleq \sup_k |e_k|$. l^1 denotes the space of real sequences, equipped with the norm $\|q\|_1 = \sum_{k=0}^{\infty} |q_k| < \infty$. $\mathcal{L}^p(R_+)$ denotes the space of measurable functions $f(t)$ equipped with the norm $\|f\|_p = (\int_0^\infty |f(t)|^p \, dt)^{\frac{1}{p}} < \infty$. $\mathcal{RL}^1$ denotes the subset of $\mathcal{L}^1$ formed by matrices with real rational Laplace transform. Given a function $q(t) \in \mathcal{L}^1$ we will denote its Laplace transform by $Q(s) \in \mathcal{L}_\infty$, and, by a slight abuse of notation, we will denote as $\|Q(s)\|_1 \triangleq \|q(t)\|_1$. Throughout the paper we will use packed notation to represent state-space realizations, i.e.,

$$G(s) = C(sI - A)^{-1}B + D \triangleq \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).$$

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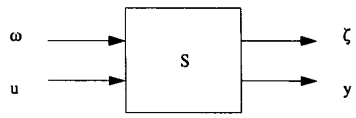

Fig. 1. The generalized plant.

Finally, given two transfer matrices $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ and Q with appropriate dimensions, the lower linear fractional transformation is defined as $\mathcal{F}_l(T, Q) \triangleq T_{11} + T_{12}Q(I - T_{22}Q)^{-1}T_{21}$.

B. The $\mathcal{L}^1$ Optimal Control Problem

Consider the system represented by Fig. 1, where S represents the system to be controlled; the scalar signals $w \in \mathcal{L}^\infty$ and u represents an exogeneous disturbance and the control action respectively and where ζ and y represent the output subject to performance constraints and the measurements available to the controller respectively. As usual we will assume, without loss of generality, that any weights have been absorbed in the plant S. Then, the $\mathcal{L}^1$ optimal control problem can be stated as: Given the system (S) find an internally stabilizing controller $u(s) = K(s)y(s)$ such that the worst case (over the set of all $w(t) \in \mathcal{L}^\infty$, $\|w\|_\infty \le 1$) maximum amplitude of the performance output $\zeta(t)$ is minimized.

C. Problem Transformation

Assume that the system S has the state-space realization

$$\left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right) \qquad (S)$$

where the pairs (A, B_2) and (C_2, A) are stabilizable and detectable respectively. It is well known (see for instance [11]) that the set of all internally stabilizing controllers can be parameterized in terms of a free parameter $Q \in \mathcal{H}_\infty$ as

$$K = \mathcal{F}_l(J, Q) \qquad (1)$$

where J has the state-space realization

$$\left(\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22}R_b \\ \hline F & 0 & R_b \\ -R_c(C_2 + D_{22}F) & R_c & -R_cD_{22}R_b \end{array}\right) \qquad (J)$$

where F and L are selected such that $A + B_2F$ and $A + LC_2$ are stable and R_b and R_c are free nonsingular matrices than can be used, for instance, to obtain an inner–outer factorization. By using this parameterization, the closed-loop transfer function $T_{\zeta w}$ can be written as

$$T_{\zeta w}(s) = T_1(s) + T_2(s)Q(s) \qquad (2)$$

where $T_i(s) \in \mathcal{RH}_\infty$. Hence the $\mathcal{L}^1$ control problem can be now precisely stated as solving

$$\mu^0 = \inf_{Q \in \mathcal{H}_\infty} \|T_1 + T_2Q\|_1.$$

To guarantee that this problem is well posed, in the sequel we will assume that $T_2(s)$ does not have zeros on the $j\omega$-axis, including ∞.

Theorem 1 (Dahleh and Pearson [2]): Let $T_2(s)$ have n distinct zeros z_k in the open right-half plane and no zeros on the jw-axis. Then:

$$\begin{aligned} \mu^0 &= \inf_{K\text{stab}} \|T_1 + T_2 * Q\|_1 \\ &= \max_{\alpha_j} \left[\sum_{i=1}^{n} \alpha_i \operatorname{Re}\{T_1(z_i)\} + \sum_{i=1}^{n} \alpha_{i+n} \operatorname{Im}\{T_1(z_i)\} \right] \end{aligned} \qquad (3)$$

subject to

$$\left| \sum_{i=1}^{n} \alpha_i \operatorname{Re}\{e^{-z_i t}\} + \sum_{i=1}^{n} \alpha_{i+n} \operatorname{Im}\{e^{-z_i t}\} \right| \le 1 \ \forall t \in \mathcal{R}_+. \qquad (4)$$

Furthermore, the optimal error ϕ has the form

$$\phi = \sum_{i=0}^{m} \phi_i \delta(t - t_i), \qquad t_i \in \mathcal{R}_+, \quad m \text{ finite} \qquad (5)$$

$$\{\phi_i\} \in l^1, \quad \|\phi\|_1 = \sum_{i=0}^{m} |\phi_i|$$

and satisfies the interpolation condition

$$\Phi(z_k) = \sum_{i=0}^{m} \phi_i e^{-z_k t_i} = T_1(z_k), \qquad k = 1, \cdots, n.$$

Remark 1: From (5) it follows that the optimal compensator has a nonrational Laplace transform.

D. Existence of Suboptimal Rational Controllers

In this section we consider the problem of approximating the optimal cost μ^0 with controllers in $\mathcal{RL}^1$. First note that, without loss of generality, we can assume $t_k = (k-1)T$, $T > 0$, $k = 1, \cdots, n$. Indeed, from Theorem 9 in [2] it follows that, given $\delta > 0$, we can take T small enough and ϕ_i such that the corresponding cost μ satisfies $\mu^0 \le \mu \le \mu^0(1 + \delta)$. Define

$$f_i^\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & t \in [t_i - \frac{\epsilon}{2}, t_i + \frac{\epsilon}{2}]; \\ 0, & \text{otherwise.} \end{cases}$$

$$f^\epsilon(t) = \sum_{i=0}^{m} f_i^\epsilon \phi_i.$$

It is immediate that $f_i^\epsilon \in \mathcal{L}^1$ and, for $\epsilon \le T$

$$\|f^\epsilon\|_1 = \sup_{v \in \mathcal{L}^\infty, \ \|v\|=1} \|f^\epsilon * v\|_\infty = \|\phi\|_1.$$

Moreover, it is easily shown that for ϵ small enough there exist ϕ_i^ϵ such that $\hat{f}(t) = \sum_{i=0}^{m} \phi_i^\epsilon f_i^\epsilon(t)$ satisfies the interpolation constraints

$$\hat{F}(z_k) = T_1(z_k), \qquad k = 1, \cdots, n$$

and such that $\phi_i^\epsilon \to \phi_i$ as $\epsilon \to 0$. Finally, since the set of functions with rational Laplace transfer functions is dense in $\mathcal{L}^1$ [12] it can be shown (see Appendix A) that given $\eta > 0$ small enough, there exist a function $f^r(t) \in \mathcal{RL}^1$ such that $\|f^r(t) - f^\epsilon(t)\|_1 \le \eta$ and such that $f^r(t)$ satisfies the interpolation constraints. It follows that the suboptimal error $f^r(t)$ can then be achieved by the stabilizing rational compensator $Q(s) = (F^r(s) - T_1(s))/T_2(s)$. These results are summarized in the following lemma.

Lemma 1: Suppose that the $\mathcal{L}^1$ optimal control problem has a (nonrational) solution with optimal cost μ^0. Then, for any $\mu^r > \mu^0$ there exists a suboptimal internally stabilizing compensator $K^r \in \mathcal{RL}^1$ such that the resulting closed-loop transfer function satisfies $\|T_{\zeta w}\|_1 \le \mu^r$.

III. Problem Solution

Although Lemma 1 guarantees the existence of a suboptimal rational compensator, the proof is not constructive. In this section we address the issue of finding a suboptimal rational controller. To that effect we introduce the concepts of the EAS and of positively invariant sets [13].

A. Definitions

Definition 1: Consider the continuous time system (S). Then, the EAS is defined as the discrete time system

$$\left(\begin{array}{c|cc} I+\tau A & \tau B_1 & \tau B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & C_{21} & D_{22} \end{array}\right) \qquad (EAS)$$

where $\tau > 0$.

Remark 2: Let $T(s)$ and $T^{(EAS)}(z, \tau)$ denote the transfer matrices of (S) and its EAS respectively. From the definition it can be easily seen that $T^{(EAS)}(z, \tau) = T(s)|_{z=1+\tau s}$. Moreover, given any controller $K(s)$ we have that the corresponding closed-loop systems satisfy

$$T_{cl}(s) = \mathcal{F}_l(T(s), K(s))$$

$$T_{cl}^{(EAS)}(z,\tau) = T_{cl}(s)|_{z=1+\tau s} = \mathcal{F}_l(T^{(EAS)}(z,\tau), K^{(EAS)}(z,\tau))$$

where $K^{(EAS)}(z, \tau) = K(s)|_{z=1+\tau s}$.

Definition 2: Consider the system

$$\dot{x}(t) = Ax(t) + Bv(t) \tag{6}$$

where $x \in R^n$ and $v(t) \in \Omega \subset R^m$. A set $\Sigma \subset R^n$ is a positively invariant set of (6) if for any initial condition $x_0 \in \Sigma$ and for any $v(t)$ the corresponding trajectory $x(t, x_0, v(t)) \in \Sigma$ for all t. A similar definition holds for the case of discrete-time systems.

B. Proposed Design Method

In this section we introduce a method for finding suboptimal rational controllers yielding cost arbitrarily close to the optimal. An additional advantage of this method is that it can be used to remove the ill-posedness arising from the existence of zeros on the jw-axis. The key to establish these results is to show that i) the l^1 norm of the EAS is an upper bound of the $\mathcal{L}^1$ norm of the continuous-time system (Theorem 2) and ii) the optimal $\mathcal{L}^1$ cost is recovered when the parameter $\tau \to 0$ (Theorem 3).

Theorem 2: Consider the system

$$\dot{x} = Ax + B_1 v$$

$$\zeta = C_1 x + D_{11} v. \tag{7}$$

Assume that the corresponding (EAS)

$$x_{k+1} = (I + \tau A)x_k + \tau B_1 v_k$$

$$\zeta_k = C_1 x_k + D_{11} v_k \tag{8}$$

is asymptotically stable and such that

$$\|T_{\zeta v}^{(EAS)}\|_1 = \sup_{\substack{v \in l^\infty,\ \|v\| \le 1 \\ x_0 = 0}} \|\zeta_k\|_\infty = \mu_E(\tau).$$

Then system (7) is asymptotically stable and such that

$$\|T_{\zeta v}\|_1 = \sup_{\substack{v \in \mathcal{L}^\infty,\ \|v\| \le 1 \\ x_0 = 0}} \|\zeta(t)\|_\infty \triangleq \mu_c \le \mu_E(\tau).$$

Conversely, if (7) is asymptotically stable and $\|T_{\zeta v}\|_1 \triangleq \mu_c$ then for all $\mu > \mu_c$ there exists $\tau^* > 0$ such that for all $0 < \tau \le \tau^*$ the EAS (8) is asymptotically stable and such that $\|T_{\zeta w}^{(EAS)}\|_1 \le \mu$.

Proof: The proof of the theorem is given in Appendix B.

Theorem 3: Consider a strictly decreasing sequence $\tau_i \to 0$, and let $\mu_i = \inf_{K\text{stabilizing}} \|T_{\zeta w}^{(EAS)}\|_1$ denote the optimal l^1 cost for the closed-loop EAS. Then the sequence μ_i is nonincreasing and such that $\mu_i \to \mu^0$, the optimal $\mathcal{L}^1$ cost.

Proof: The proof will be split into two parts. First we show that the sequence μ_i is nonincreasing. To this aim, for a given τ, let $K(z, \tau)$ denote the optimal l^1 controller for the EAS and consider the controller $K_c(s) \triangleq K(z, \tau)|_{z=1+\tau s}$. Then, the corresponding closed-loop systems are related by

$$T_{\zeta v}(s) = \mathcal{F}_l(S, K_c) \triangleq \left(\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right) \qquad (S_{cl})$$

$$T_{\zeta v}^{(EAS)}(z, \tau) = \mathcal{F}_l(EAS, K) = \left(\begin{array}{c|c} I+\tau A_{cl} & \tau B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right) \qquad (EAS_{cl})$$

and, from Theorem 2, we have that S_{cl} is stable. Let $\Sigma_E(\tau)$ denote the closure of the origin-reachable domain of (EAS_{cl}) with the bounded input $\|v\| \le 1$ and define

$$Z(\epsilon) \triangleq \{x\colon \|C_{cl}x + D_{cl}v\|_\infty \le \epsilon \text{ for all } \|v\| \le 1\}$$

$$\mu_i \triangleq \min\{\epsilon > 0\colon \Sigma_E(\tau_i) \subseteq Z(\epsilon)\}. \tag{9}$$

The set $\Sigma_E(\tau_i)$ is positively invariant for the EAS. Therefore, denoting by $\partial\Sigma_E(\tau_i)$ the boundary of $\Sigma_E(\tau_i)$, we have that for all $x \in \partial\Sigma_E(\tau_i)$ and all $\|v\| \le 1$

$$(I + \tau_i A_{cl})x + \tau_i B_{cl} v \in \Sigma_E(\tau_i) \tag{10}$$

and, by convexity, for $0 < \tau_{i+1} < \tau_i$ we have

$$(I + \tau_{i+1} A_{cl})x + \tau_{i+1} B_{cl} v \in \Sigma_E(\tau_i). \tag{11}$$

Hence $\Sigma_E(\tau_i)$ is positively invariant for (8) [15], with $\tau = \tau_{i+1}$. Since $\Sigma_E(\tau_i)$ contains the origin, then it includes $\Sigma_E(\tau_{i+1})$ so $\Sigma_E(\tau_{i+1}) \subseteq \Sigma_E(\tau_i) \subseteq Z(\mu_i)$. It follows that

$$\mu_{i+1} = \min\{\epsilon\colon \Sigma(\tau_{i+1}) \subseteq Z(\epsilon)\} \le \mu_i.$$

Since μ_i is a nonincreasing sequence, bounded below by μ^0 (from Theorem 2), it follows that it has a limit $\mu^* \ge \mu^0$. From Lemma 1 we have that, given any $\epsilon > 0$, there exist a rational controller $K(s)$ that achieves an $\mathcal{L}^1$ cost μ_r such that $\mu^0 \le \mu_r \le \mu^0 + 0.5\epsilon$. Let $\mu_E(\tau)$ denote the l^1 norm of the closed-loop EAS achieved using the controller $K(z)$, $z = 1 + \tau s$. From Theorem 2, we have that there exists τ^* such that for all $\tau \le \tau^*$, $\mu_E(\tau) \le \mu_r + 0.5\epsilon \le \mu^0 + \epsilon$. Therefore, since $\mu_i = \inf_{K\text{stabilizing}} \|T_{\zeta v}^{(EAS)}(z, \tau_i)\|_1$ we have that, for all i such that $\tau_i \le \tau^*$, $\mu^0 \le \mu_i \le \mu_E(\tau_i) \le \mu^0 + \epsilon$. Hence, $\mu^* = \lim_{i\to\infty} \mu_i = \mu^0$.

Next, we recall the main result regarding the SISO discrete-time l^1 optimal control problem.

Theorem 4 (Dahler and Pearson [2]): Let $T_2(z)$ have n distinct zeros z_k outside the closed unit disk. Then

$$\mu_d^0 = \inf_{K\text{stab}} \|T_1(z) + T_2(z) * Q(z)\|_1 = \max_{\alpha_j} \left[\sum_{i=1}^{n} \alpha_i \operatorname{Re}\{T_1(z_i)\} + \sum_{i=1}^{n} \alpha_{i+n} \operatorname{Im}\{T_1(z_i)\}\right] \tag{12}$$

subject to

$$\left|\sum_{i=1}^{n} \alpha_i \operatorname{Re}\{z_i^{-k}\} + \sum_{i=1}^{n} \alpha_{i+n} \operatorname{Im}\{z_i^{-k}\}\right| \triangleq r_k \le 1 \qquad k = 0, 1, \cdots. \tag{13}$$

Furthermore, the optimal error ϕ satisfies

$$\phi_k = 0, \text{ whenever } |r_k| < 1$$

$$\begin{aligned} \phi_k r_k &\geq 0 \\ \sum_{k=0}^{\infty} |\phi_k| &= \mu_d^0 \\ \sum_{k=0}^{\infty} \phi_k z_i^{-k} &= T_1(z_i), \text{ for all } i = 1, \cdots, n. \end{aligned} \tag{14}$$

Note that from (14) it follows that the optimal compensator is rational. Since, if a rational controller $K(z)$ yielding an l^1 cost μ_E is found for (8), then $K(\tau s+1)$ internally stabilizes (7) and yields an $\mathcal{L}^1$ cost $\mu_c \leq \mu_E$, it follows that a rational compensator can be synthesized using the EAS with suitably small τ. By combining this observation with the results of Theorems 2, 3, and 4, we can state now the main result of this section.

Theorem 5: Consider the $\mathcal{L}^1$ optimal control problem for SISO continuous time-systems. A suboptimal rational solution, with cost arbitrarily close to the optimal cost, can be obtained by solving a discrete-time l^1 optimal control problem for the corresponding EAS. Moreover, if $K(z)$ denotes the optimal l^1 compensator for the EAS, the suboptimal $\mathcal{L}^1$ compensator is given by $K(\tau s + 1)$.

Remark 3: Since the optimal discrete-time closed-loop plant has all its poles at the origin, it follows that the closed-loop continuous time-system has all its poles at $s = (-1/\tau)$.

Remark 4: The transformation $1 + \tau s$ maps the imaginary axis, except the origin, outside the unit disk. Hence, our approach maps plant zeros on $(-j\infty, j\infty) - \{0\}$ outside the unit disk, providing a guaranteed cost rational continuous-time compensator in the cases in which the optimal $\mathcal{L}^1$ theory developed in [2] fails. In particular, it provides rational continuous-time compensators for strictly proper continuous-time plants which have no zeros at the origin. In this case, in view of Theorem 1, we can achieve a cost which is arbitrarily close to the infimum of the set of all costs associated with rational compensators.

IV. A Simple Example

Consider the SISO plant used in [2]

$$P(s) = \frac{s-1}{s-2}$$

and assume that the output and measurement equations are given by

$$\zeta = Pu$$

$$y = -Pu + v$$

where $v \in \mathcal{L}^{\infty}$.

The optimal $\mathcal{L}^1$ controller is given by [2]

$$K_{\mathcal{L}^1} = \frac{(s-2)(1.7071 - 4.1213e^{-0.8814s})}{(s-1)(-0.7071 + 4.1213e^{-0.8814s})} \tag{15}$$

and yields an optimal cost $\mu^0 = 5.8284$. For $\tau = 0.1$, the Youla parameterization of the EAS, with $F = -2.9091$ and $L = 0.3667$ yields

$$T_1 = \frac{176(z-1.1)}{125(1.1z-1)(1.2z-1)}$$

$$T_2 = \frac{(z-1.1)(z-1.2)}{(1.1z-1)(1.2z-1)}. \tag{16}$$

TABLE I
Cost and Closed-Loop Systems for Several Values of τ

τ	$\|T_{\zeta v}\|_1$	$\phi(s)$
cont	5.83	$1.7071 - 4.1213e^{(-0.8814)}$
0.1	6.18	$1.84 - \frac{4.34}{(1+0.1s)^9}$
0.2	6.49	$1.86 - \frac{4.63}{(1+0.2s)^5}$
0.5	7.34	$\frac{16}{7} - \frac{36}{7(1+0.5s)^2}$

Solving for the optimal l^1 compensator yields optimal cost $\mu_d = 6.184$, with the corresponding optimal Q and compensator K_{EAS} given by

$$\begin{aligned} Q(z) = {} & 2.4309 - 0.0525z^{-1} + 0.0607z^{-2} + 0.2089z^{-3} \\ & + 0.4004z^{-4} + 0.6542z^{-5} + 0.9554z^{-6} + 1.3458z^{-7} \\ & + 1.8343z^{-8} - 3.2895z^{-9} \end{aligned} \tag{17}$$

$$K_{EAS} = \mathcal{F}_l(J, Q).$$

Finally, the transformation $z = \tau s + 1$ yields the corresponding compensator for the continuous time system. These results, along with the results of several designs obtained using different values of τ, are summarized in Table I.

V. Discussion and Conclusions

A recent research effort [1]–[4] has lead to techniques for designing optimal compensators that minimize the worst case output amplitude with respect to all inputs of bounded amplitude. In the discrete-time SISO case, minimizing the l^1 norm of the closed-loop impulse response yields a rational compensator. Unfortunately, the solution to the continuous-time version of the problem is nonrational. Thus, given the difficulty of physically implementing a system with a nonrational transfer function, in most cases this theory is primarily used to furnish a performance limit for any linear feedback compensator.

In this paper, we have proposed a suboptimal design technique which enables us to compute near-optimal continuous-time rational compensators by applying the l^1 theory to the Euler forward approximating system, followed by the transformation $z = 1 + \tau s$. We have shown that the $\mathcal{L}^1$ norm of the resulting closed-loop system is upper bounded by the l^1 cost of the corresponding EAS and that the optimal $\mathcal{L}^1$ cost is recovered as the parameter $\tau \to 0$.

One appealing feature of our technique is that through the use of the simple transformation $z = \tau s + 1$, it removes the ill-posedness due to the presence of zeros on the imaginary axis (except for those at the origin). This property allows us to obtain a guaranteed cost compensator even in the cases (such as strictly proper plants) where the $\mathcal{L}^1$ theory developed in [2] is not applicable. Moreover, our results also apply to MIMO systems. Although in this case the l^1 optimal control problem leads to infinite-dimensional optimization problems, there are currently efficient methods to get approximate rational solutions by solving suitable truncated problems [16]. These methods can be combined with our approach to furnish suboptimal rational solutions to general multiblock $\mathcal{L}^1$ problems.

Finally, while the present paper was under review, a method for synthesizing suboptimal rational $\mathcal{L}^1$ controllers was proposed by Ohta *et al.*, [10]. While this approach is completely different from the approach proposed here, there seems to be strong connections between them. Noteworthy, when the parameter λ in [10], (17) is set equal to τ in (EAS), both approaches lead to a closed-loop system with all the poles located at $s = (-1/\tau)$. In fact, it is conjectured that both approaches yield the same closed-loop system.

Although consistent numerical experience supports this fact, so far no general proof is available. Research is currently being pursued in this direction.

Appendix A
Proof of Lemma 1

Consider a strictly decreasing sequence $\epsilon_j \to 0$ and define

$$\mathcal{F} \triangleq \begin{pmatrix} e^{-z_1 t_1} & \cdots & e^{-z_1 t_n} \\ \vdots & & \vdots \\ e^{-z_m t_1} & \cdots & e^{-z_m t_n} \end{pmatrix}$$

$$\mathcal{F}^{\epsilon_j} \triangleq \begin{pmatrix} F_1^{\epsilon_j}(z_1) & \cdots & F_n^{\epsilon_j}(z_1) \\ \vdots & & \vdots \\ F_1^{\epsilon_j}(z_m) & \cdots & F_n^{\epsilon_j}(z_m) \end{pmatrix} \tag{A1}$$

$$t_i = (i-1)T, \qquad n \geq m.$$

Since all z_k are distinct, T can be selected such that $e^{-z_i T} \neq e^{-z_j T}$, $i \neq j$. It follows that $\mathcal{F}$ has full row rank since it contains a Vandermonde matrix. We will show that there exists J such that $\mathcal{F}^{\epsilon_j}$ has full row rank for all $j \geq J$. Assume, to the contrary, that there exist a sequence $\mathcal{I} = \{j_1, j_2, \cdots\}$ such that for $j \in \mathcal{I}$, $\mathcal{F}^{\epsilon_j}$ does not have full row rank. Then, there exists λ^j, $\|\lambda^j\|_\infty = 1$, such that $\lambda^j \mathcal{F}^{\epsilon_j} = 0$. Thus, since $\epsilon_i \to 0$ and z_k, $k = 1, \cdots, m$ are in the open right-half plane, we have that for any $\delta > 0$ there exists J such that

$$\begin{aligned} \left|\sum_{i=0}^{m} \lambda_i^j e^{-z_i t_k}\right| &= \left|\sum_{i=0}^{m} \lambda_i^j (e^{-z_i t_k} - F_k^{\epsilon_j}(z_i))\right| \\ &\leq \sum_{i=0}^{m} \|\lambda^j\|_\infty |e^{-z_i t_k} - F_k^{\epsilon_j}(z_i)| \\ &\leq \sum_{i=0}^{m} |e^{-z_i t_k}| \left|1 - \frac{e^{\frac{z_i\epsilon_j}{2}} - e^{\frac{-z_i\epsilon_j}{2}}}{z_i\epsilon_j}\right| \\ &\leq O^3(z_i\epsilon_j) \leq \delta \ \forall j \in \mathcal{I},\ j \geq J,\ k = 1, \cdots, n. \end{aligned} \tag{A2}$$

Since $\|\lambda^j\|_\infty = 1$, the sequence λ^j has an accumulation point $\hat{\lambda}$ such that $\|\hat{\lambda}\|_\infty = 1$ and $\hat{\lambda}\mathcal{F} = 0$. But this contradicts the fact that $\mathcal{F}$ has full row rank. Hence there exist coefficients ϕ_i^ϵ such that $\hat{f}(t) = \sum_{i=1}^m \phi_i^\epsilon f_i^\epsilon(t)$ satisfies the interpolation constraints $\hat{F}(z_k) = T_1(z_k)$. Moreover, since $\lim_{\epsilon \to 0} F_k^\epsilon(z) = e^{-z t_k}$ it follows that ϕ_i^ϵ can be selected such that $\phi_i^\epsilon \to \phi_i$. Hence $\|\hat{f}\|_1 \to \|\phi\|_1$. To complete the proof consider a sequence F_i^j of rational approximations to F_i^ϵ (in the l^1 topology) and define

$$\mathcal{F}^j \triangleq \begin{pmatrix} F_1^j(z_1) & \cdots & F_n^j(z_1) \\ \vdots & & \vdots \\ F_1^j(z_m) & \cdots & F_n^j(z_m) \end{pmatrix}$$

since

$$\left|F_i^j(z_k) - F_i^\epsilon(z_k)\right| \leq \int_0^\infty \left|f_i^j(t) - f_i^\epsilon(t)\right| dt = \|f_i^j - f_i^\epsilon\|_1$$

a similar argument shows that there exist J such that $\mathcal{F}^j$ has full row rank for $j \geq J$. It follows that, for any $\eta > 0$, there exists ϕ_i^r such that $f^r(t) = \sum_{i=1}^m \phi_i^r f_i^r(t)$ satisfies $\|f^r\|_1 - \|\phi\|_1 \leq \eta$; $F^r(s)$ is rational and satisfies the interpolation constraints $F^r(z_k) = T_1(z_k)$. The suboptimal rational compensator is given by $Q(s) = (F^r(s) - T_1(s))/T_2(s)$. ◊

Appendix B
Proof of Theorem 2

Denote by Λ the set of eigenvalues of A and define $\theta(\Lambda) \triangleq \min_{\lambda \in \Lambda} 2[(-re(\lambda)/|\lambda|^2)]$. Then (8) is asymptotically stable if and only if (7) is stable and $0 < \tau < \theta(\Lambda)$. Therefore, if A is asymptotically stable, then (7) must be so. Let Σ_C and $\Sigma_E(\tau)$ denote the closures of the origin-reachable sets of (7) and (8), with $\|v\| \leq 1$. It follows that $\mu_C = \min\{\epsilon\colon \Sigma_C \subseteq Z(\epsilon)\}$ and $\mu_E = \min\{\epsilon\colon \Sigma_E(\tau) \subseteq Z(\epsilon)\}$, where $Z(\epsilon)$ is defined in (9). The set $\Sigma_E(\tau)$ is convex and positively invariant for (8) so, denoting by $\partial\Sigma_E(\tau)$ its boundary we must have that for $x \in \partial\Sigma_E(\tau)$ and for all v such that $\|v\| \leq 1$

$$(I + \tau A)x + \tau B_1 v \in \Sigma_E(\tau). \tag{B1}$$

Let $C_{\Sigma_E(\tau)}(x)$ denote the tangent cone to $\Sigma_E(\tau)$ at x. From the convexity of $\Sigma_E(\tau)$ and (B1) it follows that

$$Ax + B_1 v \in C_{\Sigma_E(\tau)}(x). \tag{B2}$$

This condition implies [15] that the set $\Sigma_E(\tau)$ is a positively invariant set for (7). Since $\Sigma_E(\tau)$ contains the origin, it follows that it must contain Σ_C. Hence $\Sigma_C \subseteq \Sigma_E(\tau) \subseteq Z(\mu_E(\tau))$ and $\mu_C \leq \mu_E(\tau)$.

To prove the second part of the theorem consider the asymptotically stable continuous time systems

$$\dot{x} = Ax + B_1 v + \delta w \tag{B3}$$

$$\dot{x} = Ax + \delta w \tag{B4}$$

where $w(t) \in \mathcal{L}^\infty$, $\|w(t)\| \leq 1$ is a fictitious disturbance and δ is a positive weighting parameter. Denote by $\Sigma_C^*(\delta)$ and $\Sigma_W(\delta)$ the closures of the respective origin-reachable sets. Then $\Sigma_C^*(\delta)$ is given by the Minkowsky sum of Σ_C and $\Sigma_W(\delta)$. Note that the asymptotic stability of A guarantees that these sets are compact.

For $\mu > \mu_c$ the set $Z(\mu)$ contains $Z(\mu_C)$ in its interior so, by an appropriate choice of δ the set $\Sigma_w(\delta)$ can be made small enough to guarantee that $\Sigma_C^*(\delta) \subseteq Z(\mu)$. To complete the proof, we show that there exists τ^* such that for any $0 < \tau \leq \tau^*$, the set $\Sigma_C^*(\delta)$ is a positively invariant set of (8). Indeed, if this is the case then, since $\Sigma_C^*(\delta)$ contains the origin, it also contains the set $\Sigma_E(\tau)$ and therefore $\Sigma_E(\tau) \subseteq Z(\mu)$. It follows that $\mu_E(\tau) \leq \mu$. The set $\Sigma_C^*(\delta)$ contains the origin in its interior since (B3) is controllable from the input w. Since $\Sigma_C^*(\delta)$ is invariant for (B3), for each $x \in \partial\Sigma_C^*(\delta)$, and for all $\|v\| \leq 1$, $\|w\| \leq 1$, the vector $Ax + B_1 v + \delta w$ belongs to the tangent cone to $\Sigma_C^*(\delta)$ at x. It follows that there exists a strictly positive τ such that

$$x + \tau(Ax + B_1 v) \in \operatorname{int}[\Sigma_C^*(\delta)], \qquad \forall \|v\| \leq 1 \tag{B5}$$

where $\operatorname{int}(\cdot)$ denotes the interior of the set. Define

$$\tau(x) = \sup\{\tau\colon x + \tau[Ax + B_1 v] \in \Sigma_C^*(\delta)\ \forall \|v\| \leq 1\}.$$

Since $\Sigma_C^*(\delta)$ is convex and $x \in \partial\Sigma_C^*(\delta)$ if (B5) holds for some $\tau > 0$, then it holds for all $0 < \tau \leq \tau(x)$ and in particular

$$\chi = x + \frac{\tau(x)}{2}[Ax + B_1 v] \in \operatorname{int}[\Sigma_C^*(\delta)] \qquad \forall \|v\| \leq 1. \tag{B6}$$

Finally, we show that $\tau(x)$ is bounded below by a positive number as x varies on the boundary of $\Sigma_C^*(\delta)$. By contradiction, assume that there exist sequences $x_k \in \partial\Sigma_C^*(\delta)$, v_k, $\|v_k\| \leq 1$ and $\tau_k > 0$, $\tau_k \to 0$, such that

$$x_k + \tau_k(Ax_k + B_1 v_k) \notin \Sigma_C^*(\delta). \tag{B7}$$

Since $\partial\Sigma_C^*(\delta)$ and $\mathcal{B} \triangleq \{v\colon \|v_k\| \leq 1\}$ are compact sets, the sequence $\{x_k, v_k\} \in \partial\Sigma_C^*(\delta) \times \mathcal{B}$ contains a subsequence converging to a point $(\underline{x}, \underline{v})$. Hence, without loss of generality we can assume that $x_k \to \underline{x}$ and $v_k \to \underline{v}$. Select κ such that $0 < \tau_k < \frac{1}{2}\tau(\underline{x})$ for $k > \kappa$. Since $\Sigma_C^*(\delta)$ is convex and $x_k \in \partial\Sigma_C^*(\delta)$, (B7) implies that

$$\chi_k = x_k + \tfrac{1}{2}\tau(\underline{x})(Ax_k + B_1 v_k) \notin \Sigma_C^*(\delta) \qquad \text{for } k > \kappa$$

which, in view of the convergence of x_k and v_k contradicts (B6). Therefore, there exists $\tau' > 0$ such that for $0 < \tau < \tau'$, (B5) holds for all $x \in \partial\Sigma_C^*(\delta)$. It follows [14] that $\Sigma_C^*(\delta)$ is a positively invariant set for (8). The proof is completed by selecting $\tau^* = \min\{\tau', \theta(\Lambda)\}$ to guarantee asymptotic stability of system (8). $\Diamond$

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Routing with Limited State Information in Queueing Systems with Blocking

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***Abstract*— We prove the optimality of routing policies with limited state information that are employed in queueing systems with finite capacities and customers arriving at arbitrary instants. Using sample path arguments we show that, on one hand, when service times are exponential and capacities are equal, the round robin policy minimizes the expected number of losses by any time *t* and at the same time, maximizes the expected number of departures by *t*, over all policies that use no state information. On the other hand, when service times are deterministic, a modified round robin policy that makes use of limited state information outperforms stochastically all dynamic policies that have richer state information, in terms of the number of losses and the number of departures by *t*.**

I. Introduction

A classical problem in the control of queues arises when routing decisions have to be made for jobs that arrive in front of a system which consists of a number of parallel queues with identical exponential servers. If the queue lengths are observed, then the 'Join the Shortest Queue' (SQ) policy has been shown to be optimal several times in the past, first by Winston [20] who proved that, in a purely Markovian system with infinite capacities, the SQ policy maximizes the discounted number of jobs that complete service by a certain time. Weber [17], Ephremides *et al.* [4], and Walrand [16] extended Winston's results to systems with general interarrival time distributions. Menich [10] and Johri [7] established the optimality of the SQ policy in systems with state-dependent service rates and Poisson arrivals. Finally, Whitt [19] called attention to the exponentiality assumption regarding the service times and presented counterexamples to demonstrate that there exist service time distributions for which it is not always optimal to join the shortest queue. All the above authors considered systems consisting of queues with infinite buffer capacity. Recently, Hordijk and Koole in [6] and Towsley *et al.* in [15] extended the optimality of the SQ policy to finite capacity queueing systems. The common theme in the existing literature is the availability of state information that is used to determine the optimal policy. In this paper we set the problem in a different perspective. In particular, we address the following question, which is fundamental from a control standpoint "What is the structure of the optimal policy when limited state information is available?" Specifically, when the queue lengths are not observed. This situation is common in various systems; for example, in communication networks, the source station typically does not have instantaneous state information regarding the remote destination stations.

The paper answers the above question in the context of systems with finite capacities. In particular, we characterize the system's performance by the following two counting processes: a) the number

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