TABLE I
$\mu_1 = 9, \mu_2 = 3$

		batches		constant	batches		geometric
K	r	m^*	M^*	g^*	m^*	M^*	g^*
0	0	2	3	55.6821	2	3	98.8299
0	10	4	5	64.8946	4	5	108.0099
0	20	6	7	73.9300	6	7	117.0334
10	0	0	5	56.5324	0	7	99.4304
10	10	0	9	65.8836	2	9	108.6205
10	20	3	11	74.8315	4	11	117.6380
20	0	0	9	57.3500	0	9	99.9185
20	10	0	10	66.4036	0	11	109.0725
20	20	2	12	75.3949	3	13	118.0956

TABLE II
$\mu_1 = 11, \mu_2 = 1$

		batches		constant	batches		geometric
K	r	m^*	M^*	g^*	m^*	M^*	g^*
0	0	6	7	55.8945	7	8	98.9602
0	10	13	14	64.6821	14	15	107.8521
0	20	20	21	72.6218	21	22	116.2109
10	0	0	11	56.4064	3	12	99.3677
10	10	8	19	65.2098	10	20	108.2550
10	20	15	26	73.1100	17	27	116.5940
20	0	0	13	56.7562	0	15	99.6517
20	10	6	21	65.5604	8	23	108.5512
20	20	13	28	73.4392	15	30	116.8768

mality of a policy π^* is provided by the optimality equation $v_s(\pi^*) = \min_{a \in A(s)} T_{\pi^*}(s, a), s \in S$. Despite the overwhelming empirical evidence of the overall average cost optimality of a hysteretic control rule, a theoretical proof of this optimality is still lacking.

Remark: Another controlled queueing system to which the embedding technique for handling an infinite state space can be successfully applied is the following one. Consider the heterogeneous server model with multiple slower servers, but without fixed switching costs. Suppose that there are K slower servers, $i = 1, \cdots, K$, with exponential service rates $\mu_1, \cdots, \mu_K$. There is one fast server who is always activated and provides service at an exponential rate of μ with $\mu > \max_i \mu_i$. There is an operating cost at a constant rate of $r_i > 0$ per unit of time the slower server i is on. Moreover, there are linear holding costs for the jobs in the system. Numerical investigations lead to the following interesting conjecture about the structure of an overall average-cost optimal policy. Assume that the slower servers are numbered such that $r_1/\mu_1 < r_2/\mu_2 < \cdots < r_K/\mu_K$. Then the optimal control rule is characterized by critical numbers $1 < m_1 < m_2 < \cdots < m_K$: the slower servers $s = 1, \cdots, p$ are used when the number of jobs in the system is between the levels m_p and m_{p+1} where $m_{K+1} = \infty$. It is still an open problem to prove the theoretical optimality of this control rule.

V. Numerical Results

We will now give some numerical results of the tailor-made policy-iteration algorithm. In all our examples we have taken $h = 1$, $\lambda = 1.3875$, and $E[B] = 8$. The load factor $\rho = \lambda E[B]/(\mu_1 + \mu_2)$ is kept constant as 0.925, and also we keep $\mu_1 + \mu_2 = 12$. The slower service rate μ_2 is varied as 1 and 3. The fixed switching cost K and the operating cost r are varied from 0 to 20 with step size 10. We consider both the case of a constant batch size and the case of a geometrically distributed batch size. For each of the examples we give the best policy $\pi^* = (m^*, M^*)$ and its corresponding average cost $g^* = g(\pi^*)$. The number of iterations per example varied between 3 and 15 and the computing time was negligible. It is remarkable that the minimal average cost is rather insensitive to the values of μ_1 and μ_2 when $\mu_1 + \mu_2$ is kept fixed.

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A Convex Optimization Approach to Fixed-Order Controller Design for Disturbance Rejection in SISO Systems

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***Abstract*—The problem of rejection of persistent unknown-but-bounded disturbances can be solved using the well-known l^1 design approach. However, in spite of its success, this theory suffers from the fact that the resulting controller may have arbitrarily high order, even in the state-feedback case. In addition, system performance is optimized under the assumption of zero initial conditions. In this paper we propose a new approach to the problem of synthesizing fixed order controllers to optimally reject persistent disturbances. The main result of the paper shows that this approach leads to a finite-dimensional convex optimization problem that can be efficiently solved.**

***Index Terms*—Convex optimization, disturbance rejection, l^1 control.**

I. Introduction

A large number of control problems can be recast as the problem of synthesizing a controller capable of stabilizing a given linear time invariant system while, at the same time, minimizing the worst case response to some exogenous disturbances. When the signals involved

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are persistent bounded signals, with size measured in terms of peak time-domain values, it leads to l^1 optimal control theory [9], [2], [3], [5] (see also [1] for earlier related work).

The l^1 theory success lies on the fact that it directly incorporates time-domain specifications. Moreover, it furnishes a complete solution to the robust performance problem [4]. However, in contrast with $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control, l^1 optimal controllers can have arbitrarily high order [6]. Moreover, this theory cannot accommodate nonzero initial conditions.

Motivated by these difficulties, in this paper we propose a new approach to synthesizing fixed order controllers for persistent disturbance rejection in SISO systems. Rather than assuming zero initial conditions, we impose a magnitude constraint on the past outputs, implicitly defining *a set of possible initial conditions* compatible with this constraint and the disturbance bound. This leads to the basic idea of the paper, the concept of *equalized performance*. In plain words a linear single-input/single-output (SISO) plant of order n achieves an equalized performance level μ if, whenever n consecutive output values have magnitude less than μ, the same condition is repeated in the future. Thus having finite equalized performance is a stronger property than stability (while having finite ℓ^1 induced norm is equivalent to asymptotic stability). Nevertheless, as we show in the sequel, finite equalized performance can be achieved by closing the loop with a controller having at least the same order of the plant.

The main results of the paper can be summarized as follows.

- The problem of finding a *fixed order* controller achieving a given equalized performance level μ leads to a linear programming problem whose dimension is known *a priori* and it does not depend on the problem data.
- The optimal value of μ (and the corresponding controller) can be computed in polynomial time.
- The proposed technique is applicable even in cases where ℓ^1 theory breaks down, such as when the plant has zeros on the stability boundary, and can be easily extended to handle parametric uncertainty.

II. The Equalized Performance Problem

A. Notation

Given a sequence $h \in \ell^1$, its λ-transform is defined as $H(\lambda) \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} h_i \lambda^i$.[1] Given a polynomial $P(\lambda) = \sum_{i=0}^{n} a_i \lambda^i$ we denote its coefficients vector as $a \stackrel{\text{def}}{=} [a_o \quad a_1 \quad \cdots \quad a_{n-1}]^T$. The vector a deprived of the leading coefficient will be denoted by $\tilde{a}$, i.e., $\tilde{a} \stackrel{\text{def}}{=} [a_1 \quad \cdots \quad a_{n-1}]^T$. The projection operator $\mathcal{P}_N\colon \ \ell^\infty \to \ell^\infty$ is defined by

$$\mathcal{P}_N\left[x^{(0)},\, x^{(1)},\, \cdots\right] \doteq \left[x^{(0)},\, x^{(1)},\, \cdots,\, x^{(N-1)},\, 0,\, 0,\, \cdots\right]. \quad (1)$$

B. Definitions and Preliminary Results

Consider a stable SISO plant defined by the following transfer function:

$$e(\lambda) = \frac{\sum_{j=0}^{n} b_j \lambda^j}{\sum_{i=0}^{n} a_i \lambda^i}\, w(\lambda), \qquad a_0 = 1. \quad (2)$$

[1]Note that this is the inverse of the usual z transform. Therefore for causal, stable systems $H(\lambda)$ is analytical in $|\lambda| < 1$.

To this plant we can associate the following ARMA model:

$$e(k) = -\sum_{i=1}^{n} a_i e(k-i) + \sum_{j=0}^{n} b_j w(k-j) \quad (3)$$

or equivalently, the set of equations

$$\begin{aligned} x_1(k+1) &= -a_1 x_1(k) + x_2(k) + (b_1 - b_0 a_1) w(k) \\ &\ \ \vdots \\ x_n(k+1) &= -a_n x_1(k) + (b_n - b_0 a_n) w(k) \\ e(k) &= x_1(k) + b_0 w(k). \end{aligned} \quad (4)$$

For any positive integer k we have that

$$e^{(k)} = \mathcal{O}^{(k)} x(0) + \mathcal{H}^{(k)} w^{(k)} \quad (5)$$

where

$$\begin{aligned} w^{(k)} &\stackrel{\text{def}}{=} [w(0) \quad w(1) \quad \cdots \quad w(k-1)]^T \\ e^{(k)} &\stackrel{\text{def}}{=} [e(0) \quad e(1) \quad \cdots \quad e(k-1)]^T \\ \mathcal{H}^{(k)} &\stackrel{\text{def}}{=} \begin{bmatrix} b_0 & 0 & 0 & \cdots & 0 \\ CB & b_0 & 0 & \cdots & 0 \\ CAB & CB & b_0 & \cdots & 0 \\ \cdots & \cdots & & \cdots & 0 \\ CA^{k-2}B & \cdots & \cdots & CB & b_0 \end{bmatrix} \\ \mathcal{O}^{(k)} &\stackrel{\text{def}}{=} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{k-1} \end{bmatrix}. \end{aligned} \quad (6)$$

Considering the above relationship in the case $k = n$ establishes a (well-known) correspondence between any minimal quadruple (A, B, C, D) and the ARMA model (3), in the following sense: Given any $w(k)$ and any initial condition $x(0)$, the corresponding output sequence $e(k)$ of the former is an admissible evolution of the latter. Conversely, any evolution of the ARMA model is an admissible output sequence for the system having the state space realization (4), for a suitable choice of the initial state $x(0)$. Since $\mathcal{O}^{(n)}$ is invertible, determining $x(0)$ is immediate. Also note that *for a given sequence* $w(k)$, *there is a one to one correspondence between the first* n *values of* $e(k)$ *and the initial condition* $x(0)$.

Next we recall the usual ℓ^1 performance definition.

Definition 1: The plant (2) has ℓ^1 performance less than μ_{ℓ^1} iff for $x_i(0) = 0$, $i = 1, \cdots n$, and for all sequences $w(k)$, $k = 0, 1, \cdots$, such that $|w(k)| \le 1$, we have $|e(k)| \le \mu_{\ell^1}$.

Motivated by this definition, we introduce now the concept of *equalized performance*.

Definition 2: A stable plant of the form (2) has (finite) equalized performance less than μ iff for $|e(i)| \le \mu$, $i = 0, \cdots, n-1$, and for $|w(j)| \le 1$, $j = 0, 1, \cdots$,

$$|e(k)| \le \mu, \qquad k \ge n. \quad (7)$$

The term *equalized* stems from the fact that the definition above is strictly equivalent to setting the first n values of $|e(k)|$ *all equal to* μ (in all possible ways) and requiring that $|e(k)| \le \mu$ in the future.

So far we have considered the case where the length of the output string coincides with the McMillan degree of the plant (in the sequel we will sometimes refer to this case as the *natural* performance case).

However, addressing some technical points such as stable pole/zero cancellations requires extending this definition to strings of length $N > n$.

Definition 3: A stable plant of the form (2) has (finite) equalized N-performance less than μ iff for $|e(i)| \leq \mu$, $i = 0, \cdots N-1$, and all sequences $w(k)$, $k = 0, 1, \cdots$, $|w(k)| \leq 1$, compatible with $e(i)$, $i = 0, \cdots, N-1$[2] we have that

$$|e(k)| \leq \mu, \qquad k \geq N. \tag{8}$$

Thus a plant achieves equalized N-performance less than μ if whenever a string of N consecutive output values $e(0), e(1), \cdots, e(N-1)$ is below the magnitude μ, then the same condition is repeated in the future, for all possible values of the exogenous disturbance w *that could have generated* the sequence of output values for some appropriate initial condition. In the special case where $n = N$, the sequences $|e(k)| \leq \mu$ and $|w(k)| \leq 1$, $k = 0, 1, \cdots, n-1$ can be chosen independently. On the other hand, if $N > n$, then the constraint $|e(i)| \leq \mu$, $i = 0, \cdots, N-1$ *imposes an additional constraint* on the first N values of the sequence $w(k)$.

The set of admissible initial conditions, i.e., the set of initial conditions that, together with an appropriately chosen sequence of disturbances, generate a sequence of N outputs having magnitude less than μ, is given by

$$\mathcal{X}^{(N)}(\mu) = \left\{ x(0)\colon \left\| \mathcal{O}^{(N)} x(0) + \mathcal{H}^{(N)} w^{(N)} \right\|_\infty \leq \mu, \text{ for some } \left\| w^{(N)} \right\|_\infty \leq 1 \right\}. \tag{9}$$

Remark 1: The set $\mathcal{X}^{(N)}(\mu)$ always contains the origin, and, if (A, C) is observable, it is a compact polyhedron. Furthermore, $\mathcal{X}^{(N')}(\mu) \subset \mathcal{X}^{(N)}(\mu)$, if $N' > N$.

Remark 2: From Definition 3 and linearity the following properties can be easily established.

1) If a plant has equalized N-performance less than μ, it also has equalized N-performance less than μ' for all $\mu' \geq \mu$.
2) If a plant has equalized N-performance ($N \geq n$) less than μ then it has equalized N'-performance less than μ for all $N' > N$.
3) Once N consecutive output values are below a given level $\mu' \geq \mu$, then $|e(k)| \leq \mu'$ for all k.

Motivated by these properties we introduce the following definition.

Definition 4: The equalized N performance level μ^N of a stable plant is defined as: $\mu^N = \inf\{\mu$: the plant has equalized N-performance less or equal than $\mu\}$.

Remark 3: It is easy to show that not all stable plants have finite equalized N-performance for a given N. However, as we show in Section III, any stable plant achieves equalized N-performance for some $\mu > 0$ provided that N is sufficiently large.

Remark 4: Since the set $\mathcal{X}^{(N)}(\mu)$ includes the origin, it follows that $\mu_{\ell^1} \leq \mu^N$. In the special case where $\mu_{\ell^1} = \mu^N$ the plant is said to be N-equalized.

III. Equalized Performance Characterization

In this section we present some properties of plants achieving a given equalized N-performance level μ. For simplicity we assume that $\|b\| \neq 0$ (the case $\|b\| = 0$ will be reconsidered later).

Next we address the issue of computing the equalized N-performance level of a given plant.

[2]In the sense that there exists an initial condition $x(0)$ such that the output corresponding to this initial condition and the sequence of inputs $w(k)$, $k = 0, \cdots, N-1$ is precisely $e(i)$, $i = 0, \cdots, N-1$.

Theorem 1: Let $\mu \geq 0$. The plant (2) has equalized n-performance less than μ iff the following condition holds:

$$\mu \|\tilde{a}\|_1 + \|b\|_1 \leq \mu. \tag{10}$$

Therefore the equalized n-performance level μ^n of the plant is given by

$$\mu^n = \frac{\|b\|_1}{1 - \|\tilde{a}\|_1}. \tag{11}$$

Proof—Necessity: Assume that (10) does not hold. From (3), we have that

$$|e(n)| = \left| -\tilde{a}^T e^{(n)} + b^T w^{(n)} \right|. \tag{12}$$

Thus, there exist $|e(k)| \leq \mu$ and $|w(k)| \leq 1$, $k = 0, 1, \cdots, n-1$, such that[3]

$$\begin{aligned} |e(n)| &= \|\tilde{a}\|_1 \left\| e^{(n)} \right\|_\infty + \|b\|_1 \left\| w^{(n)} \right\|_\infty \\ &= \mu \|\tilde{a}\|_1 + \|b\|_1 > \mu. \end{aligned}$$

Sufficiency: If (10) holds, from (12) we have $|e(n)| \leq \mu$. Since $|w(n+1)| \leq 1$, using (10) again and replacing n by $n+1$ in (12) yields $|e(n+1)| \leq \mu$. The proof follows now by induction. □

Remark 5: From Theorem 1 we have that if $b \neq 0$, a necessary condition for a plant to have finite n equalized performance is $\|\tilde{a}\|_1 < 1$. It is clear that this condition implies system stability. If $b = 0$, a necessary condition is $\|\tilde{a}\|_1 \leq 1$.

We consider now the general case where $N \geq n$. To this effect define $m = N - n$ and consider the following set of $m + 1$ equations:

$$\begin{aligned} e(n) &= \sum_{i=1}^{n} a_i e(n-i) + \sum_{j=0}^{n} b_j w(n-i) \\ &\vdots \\ e(N) &= \sum_{i=1}^{n} a_i e(N-i) + \sum_{j=0}^{n} b_j w(N-i). \end{aligned} \tag{13}$$

Eliminating $e(N-1), e(N-2), \cdots, e(n)$, yields

$$e(N) = \sum_{i=1}^{n} a_i^{(m)} e(n-i) + \sum_{j=0}^{N} b_j^{(m)} w(N-j) \tag{14}$$

where the $a_i^{(m)}$, $i = 1, 2, \cdots, n$, and the $b_j^{(m)}$, $j = 0, 1, \cdots, N$, are functions of the coefficients a_i and b_j of (3). This expression, combined with Definition 3, leads to the following result.

Theorem 2: The plant (2) has equalized N-performance less than μ if:

$$\mu \left\| \tilde{a}^{(m)} \right\|_1 + \left\| b^{(m)} \right\|_1 \leq \mu \tag{15}$$

where

$$\tilde{a}^{(m)} = \left[a_1^{(m)}, \cdots, a_n^{(m)} \right]^T, \qquad b^{(m)} = \left[b_0^{(m)}, \cdots, b_n^{(m)} \right]^T.$$

Therefore an upper bound for the equalized N-performance level of an nth-order plant is given by

$$\overline{\mu}^N = \frac{\left\| b^{(m)} \right\|_1}{1 - \|\tilde{a}^{(m)}\|_1}. \tag{16}$$

[3]Recall that the $e(k)$ and the $w(k)$ may be chosen independently.

Proof: Similar to the sufficiency part of Theorem 1 but using (14) instead of (3).

We stress the fact that this condition is only sufficient. Note that, contrary to the case where $N = n$, here necessity fails because now the sequences $e(k)$ and $w(k)$ cannot be chosen independently. Note that stability of the plant implies that as $m \rightarrow \infty$ then the coefficients $a_i^{(m)} \rightarrow 0$. This leads to the following important facts.

Corollary 1: If the plant (2) is stable, it has finite equalized N-performance less than μ for some $N = n + m$, with m sufficiently large.

Corollary 2: If the plant (2) has a finite impulse response then it is N-equalized for all $N \geq n$.

Next we establish that as N increases the equalized N-performance level μ^N approaches from above the ℓ^1 performance level.

Theorem 3: Let $\overline{\mu}^\infty \doteq \lim_{N\rightarrow\infty} \overline{\mu}^N = \lim_{m\rightarrow\infty} \overline{\mu}^{n+m}$. If the plant (2) is stable, then its N-equalized performance μ^N level approaches its ℓ^1 performance level μ_{ℓ^1} as $N \rightarrow \infty$, i.e.,

$$\overline{\mu}^\infty = \lim_{m\rightarrow\infty} \sum_{j=0}^{n+m} \left| b_j^{(m)} \right| = \mu_{l^1}. \tag{17}$$

Proof: The first equality follows from the fact that the coefficients $a_i^{(m)}$ are (finite) linear combinations of the elements of the matrix A^m and therefore, since the plant is stable, $a_i^{(m)} \rightarrow 0,\ i = 1, \cdots, n$ as $m \rightarrow 0$. To complete the proof we need to show that

$$\lim_{m\rightarrow\infty} \sum_{j=0}^{n+m} \left| b_j^{(m)} \right| = \mu_{l^1}. \tag{18}$$

Given any sequence $w(t)$, with $|w(t)| \leq 1,\ t \geq 0$, there exists an initial condition vector $\tilde{x}$ such that $e(j) = 0,\ j = 0, 1, \cdots, n-1$. From (14) the trajectory corresponding to this initial condition and the sequence $w(t)$ is given by

$$e(N) = \sum_{j=0}^{N} b_j^{(m)} w(N-j).$$

Denote now by $\hat{e}(N)$ the trajectory corresponding to the initial condition $x_i(0) = 0,\ i = 1, \cdots, n$. This trajectory is given by

$$\hat{e}(N) = \sum_{j=0}^{N} h_j w(N-j)$$

where, $h_j,\ j = 0, 1, \cdots,$ are the Markov parameters of the plant. The function $\hat{e}(k) - e(k)$ can be interpreted as a free system response (i.e., the trajectory corresponding to a zero input and initial state $\tilde{x}$). Recall that the set of all admissible initial states $\mathcal{X}(N)$ is bounded uniformly for all N [since $\mathcal{X}(N) \subset \mathcal{X}(n)$]. Therefore, given N, we have an *a priori* bound of the form $|\hat{e}(N) - e(N)| \leq K\sigma^N$, where $\rho(A) < \sigma < 1$, $\rho(A)$ denotes the spectral radius of A and K is some finite constant. Then for every sequence $|w(t)| \leq 1$ we have

$$|\hat{e}(N) - e(N)| = \left| \sum_{j=0}^{N} \left(h_j - b_j^{(m)} \right) w(N-j) \right| \leq K\sigma^N. \tag{19}$$

This fact implies that $|\|\mathcal{P}_N[h]\|_{l^1} - \|\mathcal{P}_N[b^{(N)}]\|_{l^1}| \rightarrow 0$ (we skip the details for brevity), which, together with stability of the plant establishes the desired result. □

Finally, we address the issue of equalized performance in the case where the plant realization is nonminimal. This is important in the context of synthesis because even if we start from a minimal realization, stable pole/zero cancellations may appear in the resulting closed loop system.

Theorem 4: Consider any arbitrary monic polynomial $C(\lambda)$ and assume that the ARMA model $C(\lambda)A(\lambda)y(\lambda) = C(\lambda)B(\lambda)w(\lambda)$ of order N has equalized N-performance less than μ. Then the ARMA model $A(z)y(z) = B(z)w(z)$ also has equalized N-performance less than μ.

Proof: The proof follows from the fact that $C(\lambda)A(\lambda)y(\lambda) = C(\lambda)B(\lambda)w(\lambda)$ corresponds to the ARMA model obtained by combining the equations in (13) using the coefficients of $C(\lambda)$. This model contains among its trajectories those of the reduced one. Therefore, whenever property (8) holds for the original model, it also holds for the reduced one. □

IV. Optimization of the Equalized Performance

In this section, we consider the problem of synthesizing fixed order controllers such that the resulting closed-loop optimally rejects (in the equalized performance sense) persistent disturbances. Consider a plant of the form

$$\begin{bmatrix} e(\lambda) \\ y(\lambda) \end{bmatrix} = \frac{1}{d(\lambda)} \left[\begin{array}{c|c} n_{11}(\lambda) & n_{12}(\lambda) \\ \hline n_{21}(\lambda) & n_{22}(\lambda) \end{array} \right] \begin{bmatrix} w(\lambda) \\ u(\lambda) \end{bmatrix} \tag{20}$$

where the scalar signals $u,\ w,\ y$ and e represent the control input, exogenous disturbances, measurements available to the controller and performance output, respectively. Then the optimal equalized performance problem can be precisely stated as follows.

Problem 1: Given the linear time-invariant plant (20) with McMillan degree r, find a linear time-invariant compensator of a given order $s \geq r$ such that the equalized n performance of the resulting closed-loop system is minimized, where $n = s + r$.

In the sequel we show that this problem reduces to a finite-dimensional convex optimization problem. To this effect consider a controller of the form

$$u(\lambda) = \frac{q(\lambda)}{p(\lambda)} y(\lambda) \tag{21}$$

where p is a monic polynomial of degree s. The corresponding closed-loop system is

$$e(\lambda) = \frac{n_{11}(\lambda)}{d(\lambda)} + \frac{1}{d(\lambda)} \frac{n_{12}(\lambda)q(\lambda)n_{21}(\lambda)}{[d(\lambda)p(\lambda) - n_{22}(\lambda)q(\lambda)]} w(\lambda) \tag{22}$$

where $d(s)$ is the characteristic polynomial of A. The polynomial $[n_{11}n_{22} - n_{12}n_{21}]$ has d as a factor, i.e.,

$$n_{11}(\lambda)n_{22}(\lambda) - n_{12}(\lambda)n_{21}(\lambda) = d(\lambda)\overline{n}(\lambda)$$

thus

$$\begin{aligned} &[d(\lambda)p(\lambda) - n_{22}(\lambda)q(\lambda)]e(\lambda) \\ &\quad = [p(\lambda)n_{11}(\lambda) - q(\lambda)\overline{n}(\lambda)]w(\lambda). \end{aligned} \tag{23}$$

This last expression can be rewritten as

$$d_{cl}(p, q)(\lambda)e(\lambda) = n_{cl}(p, q)(\lambda)w(\lambda). \tag{24}$$

Without loss of generality (by using an appropriate scaling if necessary), $p(\lambda)$ and $q(\lambda)$ can always be selected such that *the polynomial* $d_{cl}(p, q)(\lambda)$ has its independent term equal to one, that is

$$d_{cl}(p, q)(\lambda) = 1 + d_{cl,1}\lambda + d_{cl,2}\lambda^2 + \cdots. \tag{25}$$

This additional equality constraint guarantees both that the resulting loop is well-posed and that it has McMillan degree $n = s + r$.

From Theorem 2 it follows that the closed-loop system (24) achieves equalized performance $\leq\mu$, if and only if

$$\mu\|\tilde{d}_{cl}(p, q)\|_1 + \|n_{cl}(p, q)\|_1 \leq \mu. \tag{26}$$

Since $n_{cl}(p, q)(\lambda)$ and $\tilde{d}_{cl}(p, q)(\lambda)$ are affine functions of the coefficients of the polynomials $p(\lambda)$ and $q(\lambda)$, and since the additional constraint (25) is equivalent to a linear constraint involving only the leading coefficients of q and p, it follows that synthesizing a controller achieving a fixed, given performance level $\mu > 0$ is equivalent to finding an interior point in a convex set in the combined p, q space. Moreover, denoting by $\theta^T \doteq [p^T \quad q^T]^T$, (26) above can be written as follows:

$$\mu\|\Phi\theta\|_1 + \|\Psi\theta\|_1 \leq \mu \tag{27}$$

where Φ and Ψ are suitable matrices whose entries are functions of the plant coefficients. Thus for each candidate μ, the problem of synthesizing a controller that achieves equalized performance less than μ (or establishing that none exists) reduces to solving a feasibility problem that can be recast into a linear programming (LP) form.

Remark 6: If s, the order of the controller, is chosen to be at least as large as r, the order of the plant, this LP problem is always feasible for some μ large enough. This follows from the fact that in this case p and q can be chosen so that the corresponding closed-loop is a FIR, and thus (Corollary 2) has finite equalized performance. Moreover, since $\mu_{eq} \geq \mu_{\ell^1}$ with the equality holding for FIR plants it follows that our approach is guaranteed to yield better performance (both in the ℓ^1 and equalized senses) than the *ad-hoc* approach of forcing the closed-loop system to be an FIR and optimizing the ℓ^1 norm of its Markov parameters.

These results are summarized in the next theorem, stating the main result of the paper.

Theorem 5: Consider a system of the form (20) with McMillan degree r. Then for each $s \geq r$ there exists a compensator of the form (21) such that the resulting closed-loop system has finite equalized $(r+s)$-performance. Furthermore, given s, the problem of synthesizing a controller of order s that minimizes the equalized performance level can be solved by a globally converging procedure, entailing only the solution of a sequence of LP problems, each one having $6n + 7$ variables, $4n + 5$ inequality, and $4n + 5$ equality constraints.

Remark 7: Since both the number of constraints and variables are affine functions of n, it follows that synthesizing a controller that achieves a given equalized performance level can be solved in polynomial time. Thus, computing the optimal equalized level (within a given tolerance) can also be accomplished in polynomial time.

Note that the synthesis algorithm proposed in Theorem 5 works even if the order of the controller is selected to be smaller than r, the order of the plant. However, in this case there is no *a priori* guarantee that the problem will be feasible, even for a sufficiently large value of μ. From a practical point of view, the initial value of the controller order s_o should be selected equal to their order of the plant. This guarantees that the parametric problem will have a solution for some $\tilde{\mu}$. Once the optimal value of the equalized performance is established for this case, we can proceed, if necessary, to decrease the order of the controller as needed. This leads to a nonincreasing sequence $\mu_{opt}^{s_i} > 0$. As we show next, this sequence converges to the optimal ℓ^1 cost.

Theorem 6: Consider an increasing sequence $s_i \geq r$ and let μ_i denote the optimal equalized performance level achievable with a controller of order s_i. Assume that the plant satisfies the standard assumptions of ℓ^1 theory and let μ_{ℓ^1} denote the optimal achievable ℓ^1 performance level. Then $\mu_i \to \mu_{\ell^1}$. Moreover, there exists $\tilde{s}$ such that $\mu_i = \mu_{\ell^1}$ for all $s_i \geq \tilde{s}$.

Proof: Follows from Corollary 2 and properties of SISO optimal ℓ^1 systems. □

V. Robustness Considerations

An additional advantage of the proposed approach is that it works in the physical parameter space (rather than in the Markov parameter space). This fact renders the method less sensitive to variations in the location of poles and zeros of the plant, a problem recently brought up in the context of fragility of some control design methods [7]. In our context, model uncertainty leads to the parametric problem

$$\mu\|\Phi(q)\theta\|_1 + \|\Psi(q)\theta\|_1 \leq \mu \tag{28}$$

where $q \in Q$ is an uncertain parameter. Assuming a polytopic structure for the problem

$$\Phi(q) = \sum_{i=1}^{v} q_i\Phi_i$$
$$\Psi(q) = \sum_{i=1}^{v} q_i\Psi_i, \qquad \sum_{i=1}^{v} q_i = 1,\ q_i \geq 0,\ i = 1, \cdots, v$$

the optimization problem preserves its convex nature. Indeed (28) is equivalent to

$$\mu\|\Phi_i\theta\|_1 + \|\Psi_i\theta\|_1 \leq \mu, \qquad i = 1, \cdots, v. \tag{29}$$

To further illustrate this point, worth of further investigation, consider the problem of minimizing equalized performance of the sensitivity function corresponding to a plant $P(\lambda, \epsilon_1, \epsilon_2)$, using a fixed structure controller $C(a, b, \lambda)$, where

$$C(a, b, \lambda) = \frac{b}{1+\lambda a}, \quad \text{and} \quad P(\lambda, \epsilon_1, \epsilon_2) = \frac{1+\lambda(1-\epsilon_1)}{1-\lambda(1-\epsilon_2)}$$

where $|\epsilon_1|, |\epsilon_1| \leq \bar{\epsilon} < 1$. For arbitrarily small $\epsilon_i > 0$ the plant is open-loop stable, minimum phase. Hence, $\inf \|S\|_{\ell^1} = 0$ and performance arbitrarily close to optimal can be achieved by using a static gain $K \to \infty$. On the other hand, a simple root locus argument shows that, for $\epsilon_1 < 0$, with ϵ_1 arbitrarily small, when $K \to \infty$ the closed-loop system becomes unstable. It follows that the optimal ℓ^1 controller is fragile in the sense that arbitrarily small plant perturbations render the closed-loop system unstable. On the other hand, the equalized performance minimization problem can be solved via convex optimization by considering all the possible combinations of $\epsilon_1, \epsilon_2 \in \{-\bar{\epsilon}, \bar{\epsilon}\}$, leading us to the four conditions

$$\mu\{|a+b-(1+\epsilon_2)| + |b(1+\epsilon_1) - a(1+\epsilon_2)|\}$$
$$+ 1 + |a+(1+\epsilon_2)| + |a| \leq \mu.$$

For $\bar{\epsilon}$ sufficiently small this problem has a feasible solution which provides the robust optimal equalized performance.

VI. Example

Example 1: Consider the following third-order system, taken from [6]:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{ccc|cc} 2.7 & -23.5 & 4.6 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1 & -2.5 & 1.501 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right).$$

The optimal ℓ^1 controller has order 16. The corresponding closed-loop is an 18th order FIR, with ℓ^1 norm $\mu_{\ell^1} = 3.01$. Table I shows a comparison of this optimal ℓ^1 controller versus the optimal equalized controllers obtained by selecting different values for the controller order. In

TABLE I
CLOSED-LOOP ℓ^1 NORM FOR DIFFERENT EQUALIZED DESIGNS

controller order	16 (optimal ℓ^1)	3	4	6	8
$\|\Phi\|_{\ell^1}$	3.01	3.85	3.42	3.16	3.07

this particular example in all cases the resulting equalized controllers rendered the closed-loop system an FIR, and thus $\mu_{eq} = \mu_{\ell^1}$. Notice that by the time the order of the controller is selected to be 8, the corresponding performance is 3.07. Thus, when compared with the optimal ℓ^1 controller we have a significant order reduction (50%) at the price of about 2% increase in cost.

Note that in this case the optimal equalized closed-loop system has a finite impulse response. Numerical experiment show that in practice this is often the case, but there are some counterexamples available where this property does not hold.

VII. DISCUSSION OF THE METHOD AND CONCLUSIONS

In this section we comment on some of the features of the proposed method. In particular, we have the following.

1) Recall that in Section III we assumed that $b \neq 0$. Through Theorem 1 this guarantees that $\|\tilde{a}\| < 1$ which implies asymptotic stability. If $b = 0$, the inequality $\mu\|\tilde{a}\| \leq \mu$ requires that $\|\tilde{a}\| \leq 1$, and this property implies only marginal stability. Thus there might be trajectories that do not converge (but that do not diverge as well). Clearly, the feasible solutions p, q of (26) might render $n_{cl}(p, q) = 0$. This difficulty can be solved by replacing condition (26) by

$$\mu\|\tilde{d}_{cl}(p, q)\|_1 + \|n_{cl}(p, q)\|_1 \leq \mu - \epsilon \tag{30}$$

where ϵ is arbitrarily small. Thus if $\|n_{cl}(p, q)\|_1 = 0$ we still have $\|\tilde{d}_{cl}(p, q)\|_1 \leq 1 - \epsilon$, and *asymptotic stability is guaranteed*.

2) Since the proposed method forces the closed-loop characteristic polynomial to satisfy $\|\tilde{d}_{cl}\|_1 < 1$, it follows that the resulting controller internally stabilizes the loop. Note this does not prevent stable pole/zero cancellations. This leads to the following question: Suppose that an s-order controller has been found such that the closed-loop system achieves $(s + r)$-equalized performance μ^{s+r}. Assume that some zero pole cancellations occur so that the resulting closed loop has a minimal realization of order $n' < n = s + r$. Does this reduced plant achieve the same equalized n'-performance level? The answer is not necessarily. This should not be surprising, since the equalized performance framework does not assume zero initial condition. However, Theorem 4 guarantees that the reduced plant (of order $n' < n$) still achieves an n-equalized performance level less or equal to μ^{s+r}.

3) An important open question is the extension of the method to the MIMO case. In principle this could be accomplished by means of a vector ARMA model. Clearly, the definitions in the paper could be easily rephrased in a vector sense by requiring that for any output string $e(0), e(1), \cdots, e(n-1)$ whose element norms are all below μ, the norm of $e(n)$ is also below μ. However, the extension loses the physical meaning of the SISO equalized performance in the following sense: the first-order multivariable system

$$A = [a], \qquad B = [1 \quad 1], \qquad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad D = 0$$

could be associated to the equation

$$\begin{bmatrix} e_1(k+1) \\ e_2(k+1) \end{bmatrix} = a \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix} + \begin{bmatrix} w_1(k+1) \\ w_2(k+1) \end{bmatrix}.$$

However, it is immediately apparent that a true correspondence between this ARMA model and the original state space system does not exist, since in the former the output components are related by $e_1 = e_2$. Thus the extension of the method to the MIMO case does not appear to be trivial.

4) Additional features of our method are that it can be used even in cases where the plant has zeros on the stability boundary, where the traditional ℓ^1 methodology breaks down [8] and can be extended to handle parametric uncertainty.

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