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Necessary and Sufficient Conditions for Robust Performance of Systems with Mixed Time-Varying Gains and Structured Uncertainty

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Abstract

In this paper we consider the problem of assessing robust performance of systems under both time varying *memoryless* gains and structured dynamic uncertainty. Performance is measured in terms of the l^∞ to l^∞ induced gain and the (constructive) conditions are given in terms of the existence of a non–empty polyhedral set. Finally, we present an example comparing these conditions with the necessary conditions obtained using l^1 theory.

1. Introduction

A large number of control problems involve designing a controller capable of stabilizing a given linear system while minimizing the worst case response to some exogenous disturbances. This problem is relevant for instance for disturbance rejection, tracking and robustness to model uncertainty (see [14] and references therein). When the exogenous disturbances are persistent bounded signals, with size measured in terms of the peak time–domain values, it leads to the l^1 optimal control theory [14, 15, 5, 6]. Depending on the uncertainty characterization, several non–conservative robust performance conditions are available (see Table 1 in [7]). In particular, in the case of non–linear time varying model uncertainty with bounded l^∞ gain, the system possesses robust performance if and only if $\rho(M) \leq 1$) where ρ denote the spectral radius and M is a matrix containing the $\|.\|_1$ norm of the various transfer matrices comprising the nominal system. Finally, in [9], a similar necessary and sufficient condition (involving the infimum over time of the spectral radius of a matrix) is given for the case where both, the nominal system and the structured perturbation are time–varying.

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Although these conditions provide computationally simple means of testing for robust performance, there are cases of practical interest where they can be overly conservative. For instance, although it is clear that these conditions are sufficient when the uncertainty is restricted to *memoryless* time–varying gains, they are not necessary. This situation, is illustrated by the following simple example. Consider the system:

$$\begin{aligned} x(k+1) &= Ax(k) = (A_o + w(k)EF)x(k) \\ A_o &= 0.5 * \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}; \; E = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \\ F &= \begin{pmatrix} 1 & 0 \end{pmatrix} \; |w(k)| \leq w_{max} \end{aligned} \qquad (1)$$

where $w(k)$ represents dynamic model uncertainty. Define

$$G(z) = \left(\begin{array}{c|c} A_o & E \\ \hline F & 0 \end{array} \right)$$

It can be easily shown that $\|G(z)\|_{\mathcal{H}_\infty} = 1.4095$ and $\|G(z)\|_1 = \frac{5}{3}$. Thus, quadratic stability is guaranteed for $w_{max} \leq \frac{1}{\|G(z)\|_{\mathcal{H}_\infty}} = 0.7095$. On the other hand, l^1 theory yields the bound $w_{max} \leq \frac{1}{\|G\|_1} = 0.6$. However, if $w(k)$ is a memoryless gain, stability can be guaranteed for $|w(k)| \leq w_{max} < 1$.

The present paper is motivated by this example. In here we address the case of systems subject to model uncertainty including both memoryless gains and l^∞ to l^∞ bounded operators. The main result of the paper furnishes a computable necessary and sufficient condition for robust performance of these systems. Additionally, we show that although our systems are time-varying, the robust performance problem can be reduced to a robust stability problem for an augmented system. This is an unexpected result, since it is known [9] that for general time–varying systems these problems are not equivalent.

2. Preliminaries

2.1. Notation and Definitions

Given a set S, we denote by $int\{S\}$ its interior. Given a matrix A, we denote by A_i its i-th row. For $x \in R^n$ we define $|x|$ as the vector with components $|x_i|$. We denote the 1−norm as $\|x\|_1 \triangleq \sum_{i=0}^{n} |x_i|$ and the infinity norm as $\|x\|_\infty \triangleq \max_i |x_i|$. l^1 denotes the space of absolutely summable sequences $h = \{h_i\}$ equipped with the norm $\|h\|_1 \triangleq \sum_{k=0}^{\infty} |h_i| < \infty$. l^∞ denotes the space of bounded sequences $h = \{h_i\}$ equipped with the norm $\|h\|_\infty \doteq \sup_{k\geq 0} |h_i| < \infty$. We denote by l_p^∞ the space of bounded vector sequences $\{h(k) \in R^p\}$. In this space we define the norm $\|h\|_\infty \doteq \sup_i \|h_i(k)\|_\infty$. Assume now that $H : l_\infty^q \to l_\infty^p$ is a bounded linear operator defined by the usual convolution relation $y = H * u$. If we denote by $H(k)$ the Markov parameters of H, its induced $l_\infty^q \to l_\infty^p$ norm is given by:

$$\|H\|_1 \doteq \max_i \sum_{j=1}^{n} \|h_{ij}\|_1 \; = \; \max_i \sum_{k=o}^{\infty} \|h_i(k)\|_1$$

Similarly, if $H: l^\infty \to l^\infty$ is a time varying, linear, bounded causal operator with kernel $H(k,l)$, i.e.: $(Hu)(k) = \sum_{l=0}^{k} H(k,l)u(l)$ we define $\|H\|_1 \doteq \sup_k \sum_{l=0}^{k} |H(k,l)|$

Given a positive integer k, $S_k: l^\infty \to l^\infty$ and S_{-k} denote the right and left–shift–by–k operators respectively. Given a time varying, linear, bounded causal operator $M: l^\infty \to l^\infty$, we define $M^{(k)} \doteq S_{-k} M S_k$; i.e, it is an operator that acts exactly as M does k stages later.

2.2. Statement of the Problem

Consider a time-varying uncertain plant M, interconnected with n causal, linear time–varying perturbation blocks Δ_i, with $\|\Delta_i\|_{l^\infty \to l^\infty} \leq 1$ (see figure 1). The uncertain plant M is known to belong, at any given instant, to a polytope of plants, i.e., it has the following state–space realization:

$$\begin{aligned}
M(k) &= \begin{pmatrix} M_{11} & \dots & M_{1n} \\ \vdots & & \vdots \\ M_{n1} & \dots & M_{nn} \end{pmatrix} \\
M_{ij}(k) &= \left(\begin{array}{c|c} A(k) & B_{ij} \\ \hline C_{ij} & D_{ij} \end{array}\right) \\
A(k) &\in \mathcal{P} \doteq \sum_{i=1}^{m} \alpha_i(k) A_i \\
\alpha_i &\geq 0, \; \sum_{i=1}^{m} \alpha_i = 1
\end{aligned} \tag{2}$$

where $\alpha_i(k)$ represent memoryless time–varying gains.

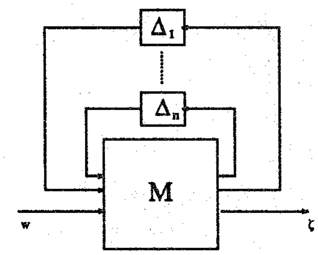

Figure 1: The Robust Performance Problem

Definition 1 *Let $\mathcal{A}$ denote the set of sequences of the form $\mathcal{A} \doteq \{A(0), A(1), \dots\}$, $A(i) \in \mathcal{P}$. The family of systems shown in figure 1, where w and z represent an l^∞ exogenous disturbance and a performance output respectively, achieves robust stability iff, for all M of the form (2) and all $\Delta \in \mathcal{D}$, $(I - M\Delta)^{-1}$ is a stable operator from $l^\infty \to l^\infty$. It achieves robust performance if, additionally, $(I - M\Delta)^{-1}$ is such that*

$$\sup_{\mathcal{A},\Delta\in\mathcal{D}} \left\{ \sup_{w\neq 0} \frac{\|\zeta\|_\infty}{\|w\|_\infty} \right\} \leq 1 \tag{3}$$

The problem that we address in this paper is to find computable, non–conservative conditions to assess whether or not the interconnection of figure 1 possesses robust performance.

Remark 1 *It can be easily shown that the problem can be recast into the form of figure 2, where the uncertainty has now the form:*

$$\Delta \in \mathcal{D} \doteq \left\{ diag(\alpha_1 I, \dots, \alpha_m I, \Delta_1, \dots, \Delta_n), \|\Delta_i\|_1 \leq 1 \right\} \tag{4}$$

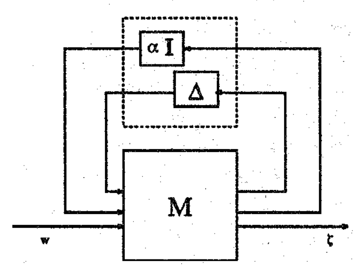

Figure 2: Robust Performance as an LFT

3. Robust performance with both operator and memoryless uncertainties

In this section we use the results of [9] on robust performance of time–varying systems to obtain necessary and sufficient conditions for the robust performance of the family (2).

3.1. Robust Stability

Lemma 1 *The interconnection (2) achieves robust stability iff there exist some integer $m > 0$ such that:*

$$\sup_{\mathcal{A}} \left\{ \sup_{k} \rho(\hat{M}_k^{(m)}) \right\} < 1 \tag{5}$$

where:

$$\begin{aligned} \hat{M}_k^{(m)} &= \begin{pmatrix} \|M_{11}^{(m)}(k_1)\|_1 & \dots & \|M_{1n}^{(m)}(k_1)\|_1 \\ \vdots & & \vdots \\ \|M_{n1}^{(m)}(k_n)\|_1 & \dots & \|M_{nn}^{(m)}(k_n)\|_1 \end{pmatrix} \\ \|M_{ij}(r)\|_1 &\doteq \sum_{l=0} |M_{ij}(r,l)| \\ k &\doteq (k_1, \dots, k_n);\ k_i \in Z^+ \end{aligned} \tag{6}$$

and where ρ denotes the spectral radius.

Proof: The proof follows immediately from applying Theorem 3 in [9] to the family (2).

In the next lemmas we show that if the time–varying gains α_i are allowed to change arbitrarily fast, then condition (5) has a simpler expression. Their proofs, omitted for space reasons, follow by exploiting the fact that the memoryless time–varying gains can change arbitrarily fast to construct suitable sequences $A = \{A(0), \dots, A(m), \dots, A(j), \dots\} \in \mathcal{A}$.

Lemma 2 *Consider the following system:*

$$\begin{aligned} M(k) &= \left(\begin{array}{c|c} A(k) & B \\ \hline C & D \end{array} \right) \\ A(k) &\in \mathcal{P} \end{aligned} \tag{7}$$

Then:

$$\sup_{\mathcal{A}} \|M(k)\|_1 = \sup_{\mathcal{A}} \|S_{-m} M(k) S_m\|_1 \ \forall\, m \in Z^+ \tag{8}$$

Lemma 3 *Consider the family (2). Then:*

$$\sup_{\mathcal{A}} \left\{ \sup_{k} \rho(\hat{M}_k^{(m)}) \right\} = \sup_{\mathcal{A}} \left\{ \sup_{k} \rho(\hat{M}_k) \right\} \tag{9}$$

where $\hat{M}_k^{(m)}$ is defined in (6) and $\hat{M}_k \doteq \hat{M}_k^{(0)}$

Lemma 4 *Let $\hat{M}_\infty \triangleq \sup_{k\to\infty} \hat{M}_k$. Then:*

$$\sup_{\mathcal{A}} \left\{ \sup_{k} \rho(\hat{M}_k) \right\} = \sup_{\mathcal{A}} \left\{ \rho(\hat{M}_\infty) \right\} \tag{10}$$

Combining the results of Lemmas 1–4 yields the main result of this section:

Theorem 1 *The interconnection of figure 1 achieves robust stability iff*

$$\sup_{\mathcal{A}} \left\{ \rho(\hat{M}_\infty) \right\} < 1 \tag{11}$$

where $\hat{M}_\infty$ is defined in Lemma 4.

Finally, in the next Theorem we show that checking condition (11) entails checking the spectral radius of the matrix obtained by considering the worst case l^1 norm of each of its elements.

Theorem 2 *Define:*

$$\hat{M}(\mathcal{A}) \doteq \begin{pmatrix} \|M_{11}(\mathcal{A})\|_1 & \dots & \|M_{1n}(\mathcal{A})\|_1 \\ \vdots & & \vdots \\ \|M_{n1}(\mathcal{A})\|_1 & \dots & \|M_{nn}(\mathcal{A})\|_1 \end{pmatrix} \tag{12}$$

where:

$$\|M_{ij}(\mathcal{A})\|_1 \doteq \sup_{\mathcal{A}} \left\{ \lim_{k\to\infty} \|M_{ij}(k)\|_1 \right\} \tag{13}$$

Then:

$$\sup_{\mathcal{A}} \left\{ \rho(\hat{M}_\infty) \right\} = \rho\Big((\hat{M}(\mathcal{A}) \Big) \tag{14}$$

In order to prove this Theorem we need the following preliminary result.

Lemma 5 *Assume that the system M is l^∞–stable for all sequences in $\mathcal{A}$ and consider any two transfer matrices M_{ij}, M_{rs}. Then, given $\epsilon > 0$ there exists a sequence $\hat{A} \in \mathcal{A}$ such that*

$$\begin{aligned} \|M_{ij}(k, \hat{A})\|_1 &\geq \sup_{A\in\mathcal{A}} \|M_{ij}(k, A)\|_1 - \epsilon \\ &\text{and} \\ \|M_{rs}(k, \hat{A})\|_1 &\geq \sup_{A\in\mathcal{A}} \|M_{rs}(k, A)\|_1 - \epsilon \end{aligned} \tag{15}$$

Proof of Theorem 2: The proof follows now from Lemma 5 by recalling that, for a positive matrix, the spectral radius is a continuous, monotonically increasing function of its elements.

3.2. Robust Performance

It is well known that [7], for Linear Time Invariant systems, robust performance is equivalent to robust stability of an *augmented* system, where the performance outputs are connected to the performance inputs via fictitious perturbation blocks. However, this equivalence breaks–down for general Time Varying systems, although it still holds in the special case of periodical systems (see Theorem 5, [9]). In the sequel we will show that the equivalence also holds for the type of systems considered here. The main idea of the proof, omitted for space reasons, follows from noticing that there exists periodic sequences which achieve costs arbitrarily close to the worst case one.

Theorem 3 *Consider the systems shown in Fig. 3, where M is of the form (2). System I achieves robust performance iff System II achieves robust stability.*

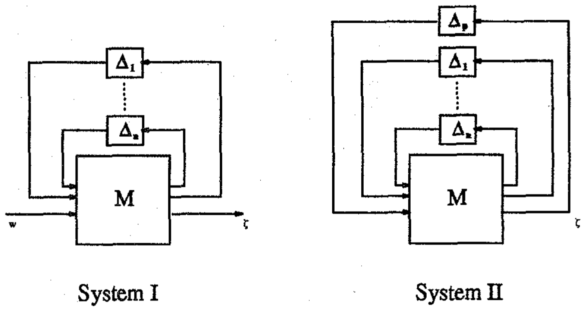

Figure 3: Equivalence Between Robust Stability and Robust Performance

4. Computation of the worst case l^1 norm

From the previous section it follows that in order to check robust stability of (2) it is necessary to compute the worst–case (over all possible sequences in $\mathcal{A}$) of an l^1—norm. In this section we furnish a procedure to compute this worst–case norm. Following the spirit of [1], this procedure is based upon the construction of an invariant set as an intersection of a sequence of polyhedral sets. In order to guarantee stability of the family of systems we need some additional assumptions.

Assumption 1 *There exists $\bar{A} \in \mathcal{P}$ such that $(\bar{A}, B)$ is reachable and $(C, \bar{A})$ is observable.*

Let $\mu > 0$ be a real parameter. Consider the set

$$\Theta(\mu) = \{x \in R^n : \|Cx - Dw\|_\infty \leq \gamma, \; \forall w \text{ s.t. } \|w\|_\infty \leq \mu\}. \tag{16}$$

It is immediate that this polyhedral set can be written in the form

$$\Theta(\mu) = \{x \in R^n : |C_i x| \leq 1 - \mu\|D_i\|_1 \quad i = 1, \ldots, p\}$$

where C_i and D_i, $i = 1, \ldots, p$ represent the $i-th$ rows and C and D respectively. Starting from this polyhedral set we can generate a proper sequence of sets proceeding backwards in time. Consider a polyhedral set

$$S = \{x : |F_i x| \leq g_i, i = 1, \ldots, s\}, \tag{17}$$

and define its preimage as the set of all state vectors x that are mapped into S by $Ax + Bw$, for all $A \in \mathcal{P}$ and all w, $\|w\| \leq \mu$, i.e.:

$$\mathcal{Q}(S) = \{x : Ax + Dw \in S, \;\; \forall \, \|w\| \leq \mu, A \in \mathcal{P}\}.$$

Since $\mathcal{P}$ is a polytope, this preimage set can be written in the form

$$\mathcal{Q}(S) = \{x : |F_i A_j x| \leq 1 - \mu\|F_i B\|_1, \quad i = 1, \ldots, s, \; j = 1, \ldots, m\}$$

Therefore the preimage set of a polyhedron is a polyhedron whose representation is furnished by the expression above. It is clear that the representation above is not minimal. As we will see in the sequel, computation of a minimal representation via elimination of redundant constraints is highly desirable. Define now the following sequence of sets:

$$\Theta^0(\mu) = \Theta(\mu)$$
$$\Theta^k(\mu) = \mathcal{Q}(\Theta^{k-1}(\mu)) \tag{18}$$

and, for $k = 0, 1, \ldots, \infty$ define the set:

$$\Omega^k(\mu) = \bigcap_{h=0}^{k} \Theta^h(\mu) \tag{19}$$

Then, the following theorem holds.

Theorem 4 *If $\|M(\mathcal{A})\|_1 < \frac{1}{\mu}$ then there exists $\bar{k}$ such that*

$$\Omega^k(\mu) = \Omega^{\bar{k}}(\mu) = \Omega^\infty(\mu), \quad k \geq \bar{k}. \tag{20}$$

If $\|M(\mathcal{A})\|_1 > \frac{1}{\mu}$ then there exists $\bar{k}$ such that

$$\Omega^k(\mu) = \emptyset. \tag{21}$$

Proof. The proof, omitted for space reasons, is based on the following fact. Denote by $x(t, A(\cdot), w(\cdot))$ the solution of $x(t+1) = A(t)x(t) + Bw(t)$ with zero initial condition and denote by $\mathcal{R}(\mathcal{P}, \mu)$ the 0-reachable state set:

$$\mathcal{R}(\mathcal{P}, \mu) = \{x = x(t, A(\cdot), w(\cdot)) \text{ for some } t > 0, \; A(k) \in \mathcal{A} \textit{ and } \|w(t)\| \leq \mu\}. \tag{22}$$

Then we have that $\|M(\mathcal{A})\|_1 < \frac{1}{\mu}$ if and only if $\mathcal{R}(\mathcal{P}, \mu) \subset int\{\Theta(\mu)\}$.

The theorem above suggests a procedure to compute upper and lower bounds for the l^1 norm of $(A(\alpha), B, C, D)$ by checking whether or not the set $\Theta^\infty(\mu)$ is empty. In practice this is accomplished by fixing a value of $\mu > 0$ and computing the sequence $\Omega^k(\mu)$ by intersecting the set $\Omega^{k-1}(\mu)$ with $\Theta^k(\mu)$. If the first condition of the theorem is satisfied for some $\bar{k}$, then we get an upper bound for the l^1 norm. Likewise, if it fails we get a lower bound. In the first case we have to reduce the valued μ (for instance halving it); in the second, to increase it (for instance doubling it). A question that arises naturally is what happens when $\mu = \|M(\mathcal{A})\|_1$. In this case, condition (20) may still be satisfied (see [2]). There are examples in which the set $\Omega^\infty(\mu)$ is not empty but does not coincide with any of the sets $\Omega^k(\mu)$.

5. A Simple Example

Consider the following simple example of a two output two input system.

$$A = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{23}$$

Assume that the uncertainties have a block diagonal structure

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \tag{24}$$

If we assume that Δ_i, $i = 1, 2$ are operators subject to $\|\Delta_i\|_1 \leq \xi$ then according to standard l^1 theory, we have to consider the following spectral radius:

$$\rho\left(\begin{bmatrix} \|M_{11}\|_1 & \|M_{12}\|_1 \\ \|M_{21}\|_1 & \|M_{22}\|_1 \end{bmatrix}\right) = \rho\left(\frac{1}{3}\begin{bmatrix} 7 & 5 \\ 5 & 7 \end{bmatrix}\right) = 4. \tag{25}$$

Therefore the stability limit is $\xi < 0.25$ as long as Δ_1 and Δ_2 are both assumed to be operators. Consider now the case in which Δ_2 is a memoryless time varying parameter. To solve the stability problem for the interconnection we have to compute the induced norm of the system having the following state–space realization:

$$(A + \Delta_2 B_2 C_2, B_1, C_1)$$

with $\|\Delta_1\|_1 \leq \mu$ and $|\Delta_2| \leq \nu$. Fixing $\mu = 0.25$ and increasing ν until the limit $\|M\mathcal{A}\|_1 \leq 4$ is violated yields approximately $\nu_{MAX} \in [0.305, 0.306]$. Indeed for $\nu = 0.306$, $\Omega^{36}(0.25)$ was found to be empty while for $\nu = 0.305$, it turns out that $\Omega^{14}(0.25) = \Omega^{13}(0.25) = \Omega^{\infty}(0.25)$. This means that the induced norm $\|M(\mathcal{A})\|_1 \in [3.268, 3.278]$. Such a set is defined (as in (17)) by 13 symmetric inequalities. Fixing now $\nu = 0.25$, and computing iteratively the maximum value of μ such that $\Omega^{\infty}(\mu)$ in not empty yields $\mu_{MAX} \in [0.291, 0.292]$. In this case,$\Omega^{34}(0.292)$ was found to be empty, while for $\nu = 0.291$ the condition $\Omega^{\infty}(0.291) = \Omega^{13}(0.291)$ was satisfied. In this case the set is defined by by 14 inequalities.

Finally, we wish to consider briefly the case where only time-varying gains are present in the system. This situation can be handled via Lyapunov methods [16][11]. In principle Assumption 1 requires the existence of operator uncertainties. Nevertheless, we can consider this problem as a "limit case". For instance if we take in this example ν small, we recover the example in the introduction. Setting $\nu = 0.0001$, we proved stability for $\mu < 0.998$ (using arguments from [1] it can indeed be shown that the actual stability condition is $\mu < \mu_{max} = 1$).

6. Conclusions

In this paper we address the problem of the analysis of robust performance and robust stability for systems containing both time–varying uncertain memoryless parameters and neglected dynamics. We prove that the problem of the robust stability can be reduced to the computation of the spectral radius of a certain non-negative matrix whose entries are the worst–case peak to peak induced norms. This worst–case analysis norm can be computed by generating sequences of sets that provide upper and a lower bounds. Finally, we show that the problem of the robust performance analysis can be reduced to a robust stability one, in a fashion similar to the time-invariant case. This fact is remarkable since it is known that this result does not hold for general time-varying systems.

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