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Rational $\mathcal{L}^1$ Suboptimal Compensators for Continuous–Time Systems

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Abstract

The persistent disturbance rejection problem ($\mathcal{L}^1$ Optimal Control) for continuous time–systems leads to non–rational compensators, even for SISO systems [1–3]. As noted in [2], the difficulty of physically implementing these controllers suggest that the most significant applications of the continuous time $\mathcal{L}^1$ theory is to furnish bounds for the achievable performance of discrete–time controllers. However, at the present time, there are no theoretical results relating the optimal l^1 norm of the discrete time system with the actual performance obtained when the controller is used in the continuous–time system. In this paper we use the theory of positively invariant sets to provide a design procedure, based upon the use of the discrete Euler approximating system, for suboptimal rational $\mathcal{L}^1$ controllers. The main results of the paper show that i) the $\mathcal{L}^1$ norm of the resulting continuous–time system is bounded above by the l^1 norm of the discrete–time counterpart and ii) the proposed rational compensators yield $\mathcal{L}^1$ cost arbitrarily close to the optimum, even in cases where the design procedure proposed in [2] fails due to the existence of plant zeros on the stability boundary.

1. Introduction

A large number of control problems involve designing a controller capable of stabilizing a given linear time invariant system while minimizing the worst case response to some exogenous disturbances. This problem is relevant for instance for disturbance rejection, tracking and robustness to model uncertainty (see [2] and references therein). When the exogenous disturbances are modeled as bounded energy signals and performance is measured in terms of the energy of the output, this problem leads to the well known $\mathcal{H}_\infty$ theory. The case where the signals involved are persistent bounded signals leads to the $\mathcal{L}^1$ optimal control theory, formulated and further explored by Vidyasagar [1, 3] and solved by Dahleh and Pearson both in the discrete [4] and continuous time [2] cases.

The $\mathcal{L}^1$ theory is appealing because it directly incorporates time–domain specifications. Moreover, it furnishes a complete solution to the robust performance problem [5]. However, in contrast with the discrete time l^1 theory, the solution to the continuous–time $\mathcal{L}^1$ optimal control problem leads to non–rational compensators, even for SISO systems. As noted in [2], the difficulty of physically implementing these controllers suggest that the most significant applications of the continuous time $\mathcal{L}^1$ theory is to provide bounds for the achievable performance of discrete–time controllers. In [6] a controller for a constrained continuous–time system was designed by first discretizing the system and then using l^1 techniques. However, at the present time, there is no theory relating the optimal value of the l^1 norm of the discretized system with the actual performance obtained when the discrete–time controller is implemented in the original continuous–time system.

In this paper we use the theory of positively invariant sets to provide a design procedure, based upon the use of the discrete Euler approximating system (EAS), for suboptimal rational $\mathcal{L}^1$ controllers. The main results of the paper show that i) the $\mathcal{L}^1$ norm of the resulting continuous–time system is bounded above by the l^1 norm of the discrete–time counterpart and ii) the optimal $\mathcal{L}^1$ system can be approximated arbitrarily close by a rational compensator related to the optimal l^1 compensator for the EAS.

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The paper is organized as follows: In section II we introduce the notation to be used and we restate the main results concerning the $\mathcal{L}^1$ problem. In section III we introduce the discrete time Euler approximating system and we propose a method for designing suboptimal rational controllers, yielding cost arbitrarily close to the optimal $\mathcal{L}^1$ cost, based upon the use of the optimal l^1 theory for the EAS. In section IV we present a simple design example and we compare our controller to the optimal $\mathcal{L}^1$ controller. Finally, in section V, we summarize our results.

2. Preliminaries

2.1 Notation

By $\mathcal{L}_\infty$ we denote the Lebesgue space of complex valued transfer matrices which are essentially bounded on the imaginary axis with norm $\|T(z)\|_\infty \triangleq \sigma_{\max}(T(jw))$.$\mathcal{H}_\infty$ denotes the set of stable complex matrices $G(s) \in \mathcal{L}_\infty$, i.e analytic in $\Re(s) \geq 0$. $\mathcal{RH}_\infty$ denotes the subset of $\mathcal{H}_\infty$formed by real rational transfer matrices. l_∞ denotes the space of bounded real sequences $\{e_k\}$ equipped with the norm $\|e\|_\infty \triangleq \sup_k |e_k|$. l^1 denotes the space of real sequences, equipped with the norm $\|q\|_1 = \sum_{k=0}^{\infty} |q_k| < \infty$. $\mathcal{L}^p(R_+)$ denotes the space of measurable functions $f(t)$ equipped with the norm: $\|f\|_p = \left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}} < \infty$. $\mathcal{RL}^1$ denotes the subset of $\mathcal{L}^1$ formed by matrices with real rational Laplace Transform. Given a function $q(t) \in \mathcal{L}^1$ we will denote its Laplace transform by $Q(s) \in \mathcal{L}_\infty$. Throughout the paper we will use packed notation to represent state–space realizations, i.e.

$$G(s) = C(sI - A)^{-1}B + D \triangleq \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

Finally, given two transfer matrices $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ and Q with appropriate dimensions, the lower *linear fractional transformation* is defined as:

$$\mathcal{F}_l(T,Q) \triangleq T_{11} + T_{12}Q(I - T_{22}Q)^{-1}T_{21}$$

2.2 The $\mathcal{L}^1$ Optimal Control Problem

Consider the system represented by the block diagram 1, where S represents the system to be controlled; the scalar signals $w \in \mathcal{L}^\infty$ and u represent an exogenous disturbance and the control action respectively; and where ζ and y represent the output subject to performance constraints and the measurements available to the controller respectively. As usual we will assume, without loss of generality, that any weights have been absorbed in the plant S. Then, the $\mathcal{L}^1$ optimal control problem can be stated as: Given the system (S) find an internally stabilizing controller

$$u(s) = K(s)y(s) \qquad (C)$$

such that the worst case (over the set of all $w(t) \in \mathcal{L}^\infty$, $\|w\|_\infty \leq 1$) maximum amplitude of the performance output $\hat{z}(t)$ is minimized.

2.3 Problem Transformation

Assume that the system S has the following state–space realization:

$$\left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right) \qquad (S)$$

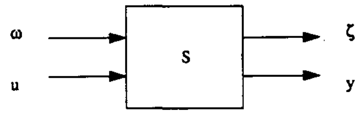

Fig. 1. The Generalized Plant

where the pairs (A, B_2) and (C_2, A) are stabilizable and detectable respectively. It is well known (see for instance [7]) that the set of all internally stabilizing controllers can be parametrized in terms of a free parameter $Q \in \mathcal{H}_\infty$ as:

$$K = \mathcal{F}_l(J, Q) \tag{1}$$

where J has the following state–space realization:

$$\left(\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22}R_b \\ \hline F & 0 & R_b \\ -R_c(C_2 + D_{22}F) & R_c & -R_cD_{22}R_b \end{array}\right) \tag{J}$$

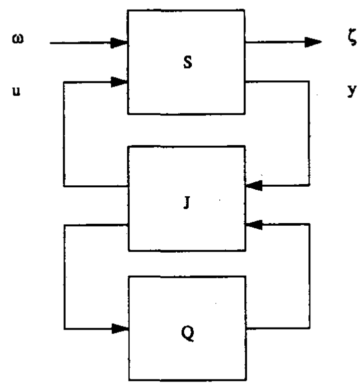

Figure 2. Parametrization of all Stabilizing Controllers

where F and L are selected such that $A + B_2F$ and $A + LC_2$ are stable and R_b and R_c are free non-singular matrices than can be used, for instance, to obtain an inner–outer factorization. By using this parametrization, the closed–loop transfer function $T_{\zeta w}$ can be written as :

$$T_{\zeta w} = \mathcal{F}_l(T, Q) = T_{11} + T_{12}QT_{21} \tag{2}$$

where $T_i \in \mathcal{RH}_\infty$ and where T has the following state–space realization:

$$T_f = \left(\begin{array}{cc|cc} A_F & -B_2F & B_1 & B_2R_b \\ 0 & A_L & B_1 + LD_{21} & 0 \\ \hline C_f + D_{12}F & -D_{12}F & D_{11} & D_{12}R_b \\ 0 & R_cC_2 & R_cD_{21} & 0 \end{array}\right) \tag{3}$$

$$A_F = A + B_2F$$
$$A_L = A + LC_2$$

For the SISO case, equation (2) reduces to:

$$T_{\zeta w}(s) = T_1(s) + T_2(s)Q(s) \tag{4}$$

where $T_2(s) = T_{12}(s)T_{21}(s)$. Finally, assume that the following conditions hold:

A1) D_{12} has full column rank and D_{21} has full row rank.

A2) $\begin{pmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{pmatrix}$ has full column rank for all ω

A3) $\begin{pmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{pmatrix}$ has full row rank for all ω.

These assumptions guarantee that the problem is well–posed: (A1) guarantees that T_{12} and T_{21} have full rank at infinity, while (A2) and (A3) rule–out the existence of zeros on the $j\omega$–axis. By using this parametrization the $\mathcal{L}^1$ control problem can be now precisely stated as solving:

$$\mu^o = \inf_{Q \in \mathcal{H}_\infty} \|T_1 + T_2 * Q\|_1$$

where $*$ denotes convolution and where $T_2(s)$ does not have zeros on the $j\omega$–axis.

• **Theorem 1:** Dahleh and Pearson, [2]Let $T_2(s)$ have n distinct zeros z_k in the open right–half plane and no zeros on the $j\omega$–axis. Then:

$$\begin{aligned} \mu^o &= \inf_{K stab} \|T_1 + T_2 * Q\|_1 \\ &= \max_{\alpha_j} \left[\sum_{i=1}^{n} \alpha_i Re\{T_1(z_i)\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{T_1(z_i)\} \right] \end{aligned} \tag{5}$$

subject to:

$$\left| \sum_{i=1}^{n} \alpha_i Re\{e^{-z_i t}\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{e^{-z_i t}\} \right| \le 1 \; \forall t \in \mathcal{R}_+ \tag{6}$$

Furthermore, the optimal error ϕ has the following form:

$$\begin{aligned} &\phi = \sum_{i=0}^{m} \phi_i \delta(t - t_i), \; t_i \in \mathcal{R}_+, \;\; \text{m finite} \\ &\{\phi_i\} \in l^1, \; \|\phi\|_1 = \sum_{i=0}^{m} |\phi_i| \end{aligned} \tag{7}$$

and satisfies the interpolation condition:

$$\Phi(z_k) = \sum_{i=0}^{m} \phi_i e^{-z_k t_i} = T_1(z_k), \; k = 1, \ldots, n$$

Remark 1: From (7) it follows that the optimal Q, and hence the optimal compensator K, have non–rational Laplace transforms.

2.4 Existence of Suboptimal Rational Controllers

In this section we consider the problem of approximating the optimal cost μ^o with controllers in $\mathcal{RL}^1$. First note that, without loss of generality, we can assume $t_k = (k-1)T$, $T > 0, k = 1, \ldots, n$. Indeed, from Theorem 9 in [2] it follows that, given $\delta > 0$, we can take T small enough and ϕ_i such that the corresponding cost μ satisfies $\mu^o \le \mu \le \mu^o(1 + \delta)$. Define:

$$f_i^\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & t \in [t_i - \frac{\epsilon}{2}, t_i + \frac{\epsilon}{2}]; \\ 0, & \text{otherwise.} \end{cases}$$

$$f^\epsilon(t) = \sum_{i=0}^{m} f_i^\epsilon \phi_i$$

It is immediate that $f_i^\epsilon \in \mathcal{L}^1$ and, for $\epsilon \le T$,

$$\|f^\epsilon\|_1 = \sup_{v \in \mathcal{L}^\infty, \|v\|=1} \|f^\epsilon * v\|_\infty = \|\phi\|_1$$

Moreover, it is easily shown that for ϵ small enough there exist ϕ_i^ϵ such that $\hat{f}(t) = \sum_{i=0}^{m} \phi_i^\epsilon f_i^\epsilon(t)$ satisfies the interpolation constraints:

$$\hat{F}(z_k) = T_1(z_k) \; k = 1, \ldots n$$

and such that $\phi_i^\epsilon \to \phi_i$ as $\epsilon \to 0$. Finally, since the set of functions with rational Laplace transfer functions is dense in $\mathcal{L}^1$ [8] it can be shown (see Appendix A) that given $\eta > 0$ small enough, there exist a function $f^r(t) \in \mathcal{RL}^1$ such that $\|f^r(t) - f^\epsilon(t)\|_1 \le \eta$ and such that $f^r(t)$ satisfies the interpolation constraints. It follows that the suboptimal error $f^r(t)$ can then be achieved by the stabilizing rational compensator $Q(s) = \frac{F^r(s) - T_1(s)}{T_2(s)}$. These results are summarized in the following lemma:

• **Lemma 1:** Suppose that the $\mathcal{L}^1$ optimal control problem has a (non rational) solution with optimal cost μ^o. Then, for any $\mu^r > \mu^o$ there exist a suboptimal internally stabilizing compensator $K^r \in \mathcal{RL}^1$ such that the resulting closed loop transfer function satisfies $\|T_{\zeta w}\|_1 \le \mu^r$.

3. Problem Solution

Although Lemma 1 guarantees the existence of a suboptimal rational compensator, the proof is not constructive. In this section we address the issue of finding a suboptimal rational controller. To that effect we introduce the concepts of the Euler Approximating System (EAS) and of positively invariant sets [9].

3.1 Definitions

• **Def. 1:** Consider the continuous time system (S). Then, the Euler Approximating System is defined as the following discrete time system:

$$\left(\begin{array}{c|cc} I+\tau A & \tau B_1 & \tau B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right) \qquad (EAS)$$

where $\tau > 0$.

• **Def. 2:** Consider the following system:

$$\dot{x}(t) = Ax(t) + Bv(t) \qquad (8)$$

where $x \in R^n$ and $v(t) \in \Omega \subset R^m$. A set $\Sigma \subset R^n$ is a *positively invariant set* of (8) if for any initial condition $x_o \in \Sigma$ and for any $v(t)$ the corresponding trajectory $x(t, x_o, v(t)) \in \Sigma$ for all t. A similar definition holds for the case of discrete–time systems.

3.2 Proposed Design Method

In this section we introduce a method for finding suboptimal rational controllers yielding cost arbitrarily close to the optimal. An additional advantage of this method is that it can be used to remove the ill-posedness arising from the existence of zeros on the jw-axis.

• **Theorem 2:** Consider the system:

$$\begin{aligned} \dot{x} &= Ax + B_1 v \\ \hat{z} &= C_1 x + D_{11} v \end{aligned} \qquad (9)$$

Assume that the corresponding (EAS):

$$\begin{aligned} x_{k+1} &= (I+\tau A)x_k + \tau B_1 v_k \\ \hat{z}_k &= C_1 x_k + D_{11} v_k \end{aligned} \qquad (10)$$

is asymptotically stable and such that:

$$\|T_{\hat{z}v}^{(EAS)}\|_1 = \sup_{\substack{v \in l^\infty,\ \|v\| \le 1 \\ x_o = 0}} \|\hat{z}_k\|_\infty = \mu_E(\tau)$$

Then the system (9) is asymptotically stable and such that:

$$\|T_{\hat{z}v}\|_1 = \sup_{\substack{v \in \mathcal{L}^\infty,\ \|v\| \le 1 \\ x_o = 0}} \|\hat{z}(t)\|_\infty \stackrel{\Delta}{=} \mu_c \le \mu_E(\tau)$$

Conversely, if (9) is asymptotically stable and $\|T_{\hat{z}v}\|_1 \stackrel{\Delta}{=} \mu_c$ then for all $\mu > \mu_c$ there exists $\tau^* > 0$ such that for all $0 < \tau \le \tau^*$ the EAS (9) is asymptotically stable and such that $\|T_{\hat{z}w}^{(EAS)}\|_1 \le \mu$.

Proof: The proof of the Theorem is given in Appendix B.

• **Theorem 3:** Consider a strictly decreasing sequence $\tau_i \to 0$, and the corresponding EAS. Let $\mu_i = \inf_{K\,stabilizing} \|T_{\hat{z}w}^{(EAS)}\|_1$ denote the optimal l^1 cost for the closed–loop system. Then the sequence μ_i is non–increasing and such that $\mu_i \to \mu^o$, the optimal $\mathcal{L}^1$ cost.

Proof: The proof will be split into two parts. First we show that the sequence μ_i is non–increasing. To this aim, let $\Sigma_E(\tau)$ denote the closure of the origin–reachable domain of (10) with the bounded input $|v| \le 1$ and define:

$$\begin{aligned} Z(\epsilon) &\stackrel{\Delta}{=} \left\{x : \|C_1 x + D_{11} v\|_\infty \le \epsilon \text{ for all } \|v\| \le 1\right\} \\ \mu_i &\stackrel{\Delta}{=} \min\{\epsilon > 0 : \Sigma_E(\tau_i) \subseteq Z(\epsilon)\} \end{aligned} \qquad (11)$$

The set $\Sigma_E(\tau_i)$ is positively invariant for the EAS. Therefore, denoting by $\partial\Sigma_E(\tau_i)$ the boundary of $\Sigma_E(\tau_i)$, we have that for all $x \in \partial\Sigma_E(\tau_i)$ and all $\|v\| \le 1$:

$$(I + \tau_i A)x + \tau_i B_1 v \in \Sigma_E(\tau_i)$$

and, by convexity, for $0 < \tau_{i+1} < \tau_i$ we have:

$$(I + \tau_{i+1} A)x + \tau_{i+1} B_1 v \in \Sigma_E(\tau_i)$$

Hence $\Sigma_E(\tau_i)$ is positively invariant for (10) [10], with $\tau = \tau_{i+1}$. Since $\Sigma_E(\tau_i)$ contains the origin, then it includes $\Sigma_E(\tau_{i+1})$ so $\Sigma_E(\tau_{i+1}) \subseteq \Sigma_E(\tau_i) \subseteq Z(\mu_i)$. It follows that:

$$\mu_{i+1} = \min\left\{\epsilon : \Sigma(\tau_{i+1}) \subseteq Z(\epsilon)\right\} \le \mu_i$$

Since μ_i is a non–increasing sequence, bounded below by μ^o (from Theorem 2), it follows that it has a limit $\mu^* \ge \mu^o$. Since from Lemma 1 we have that the optimal cost μ^o can be arbitrarily approximated with a rational controller, it follows from the second part of Theorem 2 that $\mu^* = \mu^o$ ◇.

Next, we recall the main result regarding the SISO discrete–time l^1 Optimal Control Problem:

• **Theorem 4:** Dahleh and Pearson, [2]Let $T_2(z)$ have n distinct zeros z_k outside the closed unit disk. Then:

$$\begin{aligned} \mu_d^o &= \inf_{K\,stab} \|T_1(z) + T_2(z) * Q(z)\|_1 \\ &= \max_{\alpha_j} \left[\sum_{i=1}^{n} \alpha_i Re\{T_1(z_i)\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{T_1(z_i)\}\right] \end{aligned} \qquad (12)$$

subject to:

$$\left|\sum_{i=1}^{n} \alpha_i Re\{z_i^{-k}\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{z_i^{-k}\}\right| \stackrel{\Delta}{=} r_k \le 1 \quad k = 0, 1, \ldots \qquad (13)$$

Furthermore, the optimal error ϕ satisfies:

$$\begin{aligned} &\phi_k = 0, \text{ whenever } |r_k| < 1 \\ &\phi_k r_k \ge 0 \\ &\sum_{k=0}^{\infty} |\phi_k| = \mu_d^o \\ &\sum_{k=0}^{\infty} \phi_k z_i^{-k} = T_1(z_i), \text{ for all } i = 1, \ldots, n \end{aligned} \qquad (14)$$

From (14) it follows that only finitely many ϕ_i are non–zero. Since $T_1(z)$, $T_2(z)$ are rational, it follows that the optimal compensator is also rational.

Finally, we relate the closed–loop transfer functions of (9) and its EAS (10). From the definitions it is easily seen that the closed–loop transfer function obtained by applying the rational compensator $K(s)$ to (9) is the same as the closed loop transfer function obtained by applying the compensator $K(\frac{z-1}{\tau})$ to the EAS (10) up to the complex transformation $z = \tau s + 1$. Therefore, if a rational compensator $K(z)$ yielding an l^1 cost μ_E is found for (10), then $K(\tau s + 1)$ internally stabilizes (9) and yields an $\mathcal{L}^1$ cost $\mu_c \le \mu_E$. It follows that a rational compensator can be synthesized using the *EAS* with suitably small τ. By combining this observation with the results of Theorems 2, 3 and 4, we can state now the main result of this section.

• **Theorem 5:** Consider the $\mathcal{L}^1$ Optimal Control Problem for SISO continuous time–systems. A suboptimal rational solution, with cost arbitrarily close to the optimal cost, can be obtained by solving a discrete–time l^1 optimal control problem for the corresponding EAS. Moreover, if $K(z)$ denotes the optimal l^1 compensator for the EAS, the suboptimal $\mathcal{L}^1$ compensator is given by $K(\tau s + 1)$.

Remark 2: The transformation $1+\tau s$ maps the imaginary axis, except the origin, outside the unit disk. Hence, our approach maps plant zeros on $(-j\infty,\ j\infty) - \{0\}$ outside the unit disk, providing a guaranteed cost rational continuos-time compensator in the cases in which the optimal $\mathcal{L}^1$ theory developed in [2] fails. In particular, it provides rational continuous-time compensators for strictly proper continuous-time plants which have no zeros at the origin. In this case, in view of Theorem 1, we can achieve a cost which is arbitrarily close to the infimum of the set of all costs associated with rational compensators.

4. A Simple Example

Consider the SISO plant used in [2]:

$$P(s) = \frac{s-1}{s-2}$$

and assume that the output and measurement equations are given by:

$$\begin{aligned} \hat{z} &= Pu \\ y &= -Pu + v \end{aligned}$$

where $v \in \mathcal{L}^\infty$. Then the system (S) can be represented by the following state–space realization:

$$\left(\begin{array}{c|cc} 2 & 0 & 1 \\ \hline 1 & 0 & 1 \\ -1 & 1 & -1 \end{array}\right) \qquad (S)$$

The optimal $\mathcal{L}^1$ controller is given by [2]:

$$K_{\mathcal{L}^1} = \frac{(s-2)(1.7071 - 4.1213e^{-0.8814s})}{(s-1)(-0.7071 + 4.1213e^{-0.8814s})} \qquad (15)$$

and yields and optimal cost $\mu^o = 5.8284$. For $\tau = 0.1$ the EAS is given by:

$$\left(\begin{array}{c|cc} 1.2 & 0 & 0.1 \\ \hline 1 & 0 & 1 \\ -1 & 1 & -1 \end{array}\right) \qquad (EAS)$$

and the Youla parametrization with $F = -2.9091$ and $L = 0.3667$ yields:

$$T = \left(\begin{array}{cc|cc} 0.9091 & 0.2909 & 0 & 0.0909 \\ 0 & 0.8333 & 0.3667 & 0 \\ \hline -1.9091 & 2.9091 & 0 & 0.9091 \\ 0 & -0.8333 & 0.8333 & 0 \end{array}\right)$$

$$J = \left(\begin{array}{c|cc} 1.6091 & -0.3667 & -0.2424 \\ \hline -2.9091 & 0 & 0.9091 \\ -1.5909 & 0.8333 & 0.7576 \end{array}\right)$$

Hence we have that:

$$\begin{aligned} T_1 &= \frac{176(z-1.1)}{125(1.1z-1)(1.2z-1)} \\ T_2 &= \frac{(z-1.1)(z-1.2)}{(1.1z-1)(1.2z-1)} \\ T_{\zeta w} &= T_1 + T_2 Q \end{aligned} \qquad (16)$$

Solving for the optimal l^1 compensator yields optimal cost $\mu_d = 6.184$ and optimal error:

$$\phi(z) = 1.8414 - 4.3423z^{-9}$$

The corresponding optimal Q and compensator K_{EAS} are given by:

$$\begin{aligned} Q(z) &= 2.4309 - 0.0525z^{-1} + 0.0607z^{-2} + 0.2089z^{-3} + 0.4004z^{-4} \\ &\quad + 0.6542z^{-5} + 0.9554z^{-6} + 1.3458z^{-7} + 1.8343z^{-8} - 3.2895z^{-9} \\ K_{EAS} &= \mathcal{F}_l(J, Q) \end{aligned} \qquad (17)$$

Finally, the transformation $z = \tau s + 1$ yields the corresponding compensator for the continuous time system. Although in principle the suboptimal compensator has order 10, by using model reduction techniques we were able to obtain a 4^{th} order compensator yielding a virtually identical impulse response. The closed loop system obtained with the reduced order compensator is given by:

$$T_{\zeta v} = \left(\begin{array}{ccccc|c} 0.1581 & 1.0571 & -1.0913 & 1.3769 & 0.5489 & 1.8419 \\ -1.0571 & 0.6387 & 5.7817 & 0.1740 & -0.0412 & 1.0571 \\ -1.0913 & -5.7817 & -2.3301 & 4.6435 & 1.6090 & 1.0913 \\ -1.3769 & 0.1740 & -4.6435 & -5.3510 & -6.1230 & 1.3769 \\ 0.5489 & 0.0412 & 1.6090 & 6.1230 & -9.2009 & -0.5489 \\ \hline -0.8419 & 1.0571 & -1.0913 & 1.3769 & 0.5489 & 1.8419 \end{array}\right)$$

and it is easily shown that $\|T_{\zeta w}\|_1 = |D| + \int_0^\infty |Ce^{At}B|dt = 6.184.$

5. Conclusions

A recent research effort [1–4] has lead to techniques for designing optimal compensators that minimize the worst case output amplitude with respect to all inputs of bounded amplitude. In the discrete-time SISO case, minimizing the l^1 norm of the closed-loop impulse response yields a rational compensator. Unfortunately, the solution to the continuous-time version of the problem is non–rational. Thus, given the difficulty of physically implementing a system with a non–rational transfer function, in most cases this theory is primarily used to furnish a performance limit for any linear feedback compensator.

In this paper, we have proposed a suboptimal design technique which enables to compute near-optimal *continuous-time* compensators by applying the l^1 theory to the Euler forward approximating system, hence resulting in a rational compensator. We have shown that the continuous-time cost is upper bounded by the l^1 cost and that the cost of the resulting suboptimal closed-loop system converges to the optimal one as the sampling time goes to zero.

One appealing feature of our technique is that, through the use of the simple transformation $z = \tau s + 1$, it removes the ill-posedness due to the presence of zeros on the imaginary axis (except for those at the origin). This property allows us to obtain a guaranteed cost compensator even in the cases (such as strictly proper plants) where the $\mathcal{L}^1$ theory developed in [2] is not applicable.

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Appendix A: Proof of Lemma 1

Consider a strictly decreasing sequence $\epsilon_j \to 0$ and define:

$$\mathcal{F} \triangleq \begin{pmatrix} e^{-z_1 t_1} & \dots & e^{-z_1 t_n} \\ \vdots & & \vdots \\ e^{-z_m t_1} & \dots & e^{-z_m t_n} \end{pmatrix}$$
$$\mathcal{F}^{\epsilon_j} \triangleq \begin{pmatrix} F_1^{\epsilon_j}(z_1) & \dots & F_n^{\epsilon_j}(z_1) \\ \vdots & & \vdots \\ F_1^{\epsilon_j}(z_m) & \dots & F_n^{\epsilon_j}(z_m) \end{pmatrix} \quad (A1)$$
$$t_i = (i-1)T,\ n \geq m$$

Since all z_k are distinct, T can be selected such that $e^{-z_i T} \neq e^{-z_j T}$, $i \neq j$. It follows that $\mathcal{F}$ has full row rank since it contains a Vandermonde matrix. We will show that there exists J such that $\mathcal{F}^{\epsilon_j}$ has full row rank for all $j \geq J$. Assume, to the contrary, that there exist a sequence $\mathcal{J} = \{j_1, j_2, \dots\}$ such that for $j \in \mathcal{J}$, $\mathcal{F}^{\epsilon_j}$ does not have full row rank. Then, there exists $\lambda^j, \|\lambda^j\|_\infty = 1$, such that $\lambda^j \mathcal{F}^{\epsilon_j} = 0$. Thus, since $\epsilon_i \to 0$ and z_k, $k = 1, \dots, m$ are in the open right half plane, we have that for any $\delta > 0$ there exists J such that:

$$\begin{aligned} |\sum_{i=0}^{m} \lambda_i^j e^{-z_i t_k}| &= |\sum_{i=o}^{m} \lambda_i^j (e^{-z_i t_k} - F_k^{\epsilon_j}(z_i))| \\ &\leq \sum_{i=0}^{m} \|\lambda^j\|_\infty |e^{-z_i t_k} - F_k^{\epsilon_j}(z_i)| \\ &\leq \sum_{i=o}^{m} |e^{-z_i t_k}| |1 - \frac{e^{\frac{z_i \epsilon_j}{2}} - e^{\frac{-z_i \epsilon_j}{2}}}{z_i \epsilon_j}| \\ &\leq O^3(z_i \epsilon_j) \leq \delta\ \forall j \in \mathcal{J},\ j \geq J,\ k = 1, \dots n \end{aligned} \quad (A2)$$

Since $\|\lambda^j\|_\infty = 1$, the sequence λ^j has an accumulation point $\hat{\lambda}$ such that $\|\hat{\lambda}\|_\infty = 1$ and $\hat{\lambda}\mathcal{F} = 0$. But this contradicts the fact that $\mathcal{F}$ has full row rank. Hence there exist coefficients ϕ_i^ϵ such that $\hat{f}(t) = \sum_{i=1}^{m} \phi_i^\epsilon f_i^\epsilon(t)$ satisfies the interpolation constraints $\hat{F}(z_k) = T_1(z_k)$. Moreover, since $\lim_{\epsilon \to 0} F_k^\epsilon(z) = e^{-z t_k}$ it follows that ϕ_i^ϵ can be selected such that $\phi_i^\epsilon \to \phi_i$. Hence $\|\hat{f}\|_1 \to \|\phi\|_1$. To complete the proof consider a sequence F_i^j of rational approximations to F_i^ϵ (in the l_1 topology) and define

$$\mathcal{F}^j \triangleq \begin{pmatrix} F_1^j(z_1) & \dots & F_n^j(z_1) \\ \vdots & & \vdots \\ F_1^j(z_m) & \dots & F_n^j(z_m) \end{pmatrix}$$

since:

$$|F_i^j(z_k) - F_i^\epsilon(z_k)| \leq \int_0^\infty |f_i^j(t) - f_i^\epsilon(t)| dt = \|f_i^j - f_i^\epsilon\|_1$$

a similar argument shows that there exist J such that $\mathcal{F}^j$ has full row rank for $j \geq J$. It follows that, for any $\eta > 0$, there exist ϕ_i^r such that $f^r(t) = \sum_{i=1}^{m} \phi_i^r f_i^r(t)$ satisfies $\|f^r\|_1 - \|\phi\|_1 \leq \eta$; $F^r(s)$ is rational; and satisfies the interpolation constraints $F^r(z_k) = T_1(z_k)$. The suboptimal rational compensator is given by $Q(s) = \frac{F^r(s) - T_1(s)}{T_2(s)}$ $\diamond$.

Appendix B: Proof of Theorem 2

Denote by Λ the set of eigenvalues of A and define $\theta(\Lambda) \triangleq \min_{\lambda \in \Lambda} 2[\frac{-re(\lambda)}{|\lambda|^2}]$. Then (10) is asymptotically stable if and only if (9) is stable and $0 < \tau < \theta(\Lambda)$. Therefore, if A is asymptotically stable then (9) must be so. Let Σ_C and $\Sigma_E(\tau)$ denote the closures of the origin–reachable sets of (9) and (10), with $\|v\| \leq 1$. It follows that $\mu_C = \min\{\epsilon: \Sigma_C \subseteq Z(\epsilon)\}$ and $\mu_E = \min\{\epsilon: \Sigma_E(\tau) \subseteq Z(\epsilon)\}$, where $Z(\epsilon)$ is defined in (11). The set $\Sigma_E(\tau)$ is convex and positively invariant for (10) so, denoting by $\partial\Sigma_E(\tau)$ its boundary we must have that for $x \in \partial\Sigma_E(\tau)$ and for all v such that $\|v\| \leq 1$:

$$(I + \tau A)x + \tau B_1 v \in \Sigma_E(\tau) \quad (B1)$$

Let $C_{\Sigma_E(\tau)}(x)$ denote the tangent cone to $\Sigma_E(\tau)$ at x. From the convexity of $\Sigma_E(\tau)$ and $(B1)$ it follows that:

$$Ax + B_1 v \in C_{\Sigma_E(\tau)}(x) \quad (B2)$$

This condition implies [11] that the set $\Sigma_E(\tau)$ is a positively invariant set for (9). Since $\Sigma_E(\tau)$ contains the origin, it follows that it must contain Σ_C. Hence $\Sigma_C \subseteq \Sigma_E(\tau) \subseteq Z(\mu_E(\tau))$ and $\mu_C \leq \mu_E(\tau)$.

To prove the second part of the theorem consider the asymptotically stable continuous time systems:

$$\dot{x} = Ax + B_1 v + \delta w \quad (B3)$$

$$\dot{x} = Ax + \delta w \quad (B4)$$

where $w(t) \in \mathcal{L}^\infty$, $\|w(t)\| \leq 1$ is a fictitious disturbance and δ is a positive weighting parameter. Denote by $\Sigma_C^*(\delta)$ and $\Sigma_W(\delta)$ the closures of the respective origin–reachable sets. Then $\Sigma_C^*(\delta)$ is given by the Minkowsky sum of Σ_C and $\Sigma_W(\delta)$. Note that the asymptotic stability of A guarantees that these sets are compact.

For $\mu > \mu_c$ the set $Z(\mu)$ contains $Z(\mu_C)$ in its interior so, by an appropriate choice of δ the set $\Sigma_w(\delta)$ can be made small enough to guarantee that $\Sigma_C^*(\delta) \subseteq Z(\mu)$. To complete the proof, we show that there exists τ^* such that for any $0 < \tau \leq \tau^*$, the set $\Sigma_C^*(\delta)$ is a positively invariant set of (10). Indeed, if this is the case then, since $\Sigma_C^*(\delta)$ contains the origin, it also contains the set $\Sigma_E(\tau)$ and therefore $\Sigma_E(\tau) \subseteq Z(\mu)$. It follows that $\mu_E(\tau) \leq \mu$. The set $\Sigma_C^*(\delta)$ contains the origin in its interior since (B3) is controllable from the input w. Since $\Sigma_C^*(\delta)$ is invariant for (B3), for each $x \in \partial\Sigma_C^*(\delta)$, and for all $\|v\| \leq 1$, $\|w\| \leq 1$, the vector $Ax + B_1 v + \delta w$ belongs to the tangent cone to $\Sigma_C^*(\delta)$ at x. It follows that there exists a strictly positive τ such that:

$$x + \tau(Ax + B_1 v) \in int[\Sigma_C^*(\delta)],\ \forall \|v\| \leq 1 \quad (B5)$$

where $int(.)$ denotes the interior of the set. Define:

$$\tau(x) = \sup\left\{\tau: x + \tau[Ax + B_1 v] \in \Sigma_C^*(\delta)\ \forall \|v\| \leq 1\right\}$$

Since $\Sigma_C^*(\delta)$ is convex and $x \in \partial\Sigma_C^*(\delta)$ if (B5) holds for some $\tau > 0$, then it holds for all $0 < \tau \leq \tau(x)$ and in particular:

$$\chi = x + \frac{\tau(x)}{2}[Ax + B_1 v] \in int[\Sigma_C^*(\delta)]\ \forall \|v\| \leq 1 \quad (B6)$$

Finally, we show that $\tau(x)$ is bounded below by a positive number as x varies on the boundary of $\Sigma_C^*(\delta)$. By contradiction, assume that there exist sequences $x_k \in \partial\Sigma_C^*(\delta)$, v_k, $\|v_k\| \leq 1$ and $\tau_k > 0, \tau_k \to 0$, such that :

$$x_k + \tau_k(Ax_k + B_1 v_k) \notin \Sigma_C^*(\delta) \quad (B7)$$

Since $\partial\Sigma_C^*(\delta)$ and $\mathcal{B} \triangleq \{v: \|v_k\| \leq 1\}$ are compact sets, the sequence $\{x_k, v_k\} \in \partial\Sigma_C^*(\delta) \times \mathcal{B}$ contains a subsequence converging to a point $(\underline{x}, \underline{v})$. Hence, without loss of generality we can assume that $x_k \to \underline{x}$ and $v_k \to \underline{v}$. Select κ such that $0 < \tau_k < \frac{1}{2}\tau(\underline{x})$ for $k > \kappa$. Since $\Sigma_C^*(\delta)$ is convex and $x_k \in \partial\Sigma_C^*(\delta)$, (B7) implies that:

$$\chi_k = x_k + \frac{1}{2}\tau(\underline{x})(Ax_k + B_1 v_k) \notin \Sigma_C^*(\delta) \text{ for } k > \kappa$$

which, in view of the convergence of x_k and v_k, contradicts (B6). Therefore, there exists $\tau' > 0$ such that for $0 < \tau < \tau'$, (B5) holds for all $x \in \partial\Sigma_C^*(\delta)$. It follows [10] that $\Sigma_C^*(\delta)$ is a positively invariant set for (10).The proof is completed by selecting $\tau^* = \min\left\{\tau', \theta(\Lambda)\right\}$ to guarantee asymptotic stability of system (10) $\diamond$.