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CONTROLLABILITY OF LINEAR IMPULSE DIFFERENTIAL SYSTEMS

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ABSTRACT

We give necessary and sufficient conditions for global controllability of stationary and non-stationary linear impulse differential control systems on a fixed interval.

I. INTRODUCTION

Many evolutionary processes undergo rapid changes during their development; for instance, the variation of velocity of a rocket during the separation of a stage, the work of the heart muscle, the change in a population due to external effects and the control action in pulse frequency modulated control systems. Such processes are often mathematically modeled using impulse differential equations, i.e., a system of differential equations, together with relations defining jump conditions [1]. More precisely the system dynamics are generally described by:

$$\begin{cases} \dot{x} = f(x,t) & \text{when } h(x,t) \neq 0 \\ \Delta x = j(x,t) & \text{when } h(x,t) = 0 \end{cases}$$

where $t \in R$ is the time variable, $x \in R^n$ is the state vector, f is a map from $R^n \times R$ to R^n and j: $R^n \times R$ to R^n defines the jump condition. The system undergoes a jump Δx of "size" $j(x,t)$ whenever the point (x,t) in the extended phase space meets the hypersurface of equation $h(x,t) = 0$.

In this note, we deal exclusively with deterministic linear impulse systems with fixed instants of impulse effect. More specifically we examine systems of the form:

$$(S)\begin{cases} \dot{x} = A(t)x & t \neq t_k \\ \Delta x = x(t_k^+)-x(t_k) = B_k x(t_k) + C_k u_k & t = t_k \end{cases}$$

where $A(\cdot) \in PC(R^+,R^{nxn})$, $B_k \in R^{nxn}$, $C_k \in R^{nxm}$ and $u_k \in U \subseteq R^m$ is an m-dimensional control rector. The solution of (S) starting at (x_o,t_o) is given for $t > t_o$ by:

$$x(t) = \phi(t,t_o^+)x_o + \sum_{t_o < t_k < t} \phi(t,t_k^+)C_k u_k \qquad (1)$$

where 0 is given by:

$$\phi(t,s)=U_k(t,t_k^+)\prod_{j=k}^{i+1}(I+B_j)U_j(t_j,t_{j-1}^+)(I+B_i)U_i(t_i,s)$$

for $t_{i-1} < s \leq t_i < t_k < t \leq t_{k+1}$. Here $U_k(\cdot,\cdot)$ denotes the transition matrix of $\dot{x} = A(t)x$ on $t_{k-1} < t < t_k$.

In general terms, the controllability problem deals with the following question: If in addition to the initial state x_o at $t = t_o$, a final state x_1 at $t = T$ is prescribed, does there exist a sequence (u_k) of admissible controllers that steers the state $x(t)$ from x_o to x_1 along a solution of (S)? In this note we consider the problem of global controllability (arbitrary x_o and x_1) on a fixed interval $[t_o,T]$ and give necessary and sufficient conditions for this property to hold for both time-varying and time invariant systems.

II. TIME-VARYING SYSTEMS

We will first deal with the controllability problem using a geometric approach where we will exploit the convexity of the reachable set to characterize controllability, then we will give an alternate algebraic-analytical point of view to obtain a criterion for global controllability. The natural choices for the set of admissible controllers are balls in R^{mxk} (k fixed positive integer such that $t_o < \ldots < t_k \leq T$):

$$U_q^r=\left\{u=(u_1,u_2,\ldots u_k): u_i \in R^m \text{ and } \sum_{i=1}^{k}\|u_i\|_2^q \leq r^q\right\}$$

$$U_\infty^r=\left\{u=(u_1,u_2,\ldots u_k): u_i \in R^m \text{ and } \max_{1\leq i\leq k}\|u_i\|_2 \leq r\right\}$$

Let us define the linear operator:

$$L: U_q^r \text{ to } R^n \text{ by } L(u) = \sum_{i=1}^{k} \phi(T,t_i)C_i u_i \qquad (2)$$

from equation (1), we observe that the system is controllable with respect to x_o and x_1 if and only if $x_1-\phi(T,t_o)x_o$ belongs to the range of L. Since $L(U_q^r)$ is convex then $x \in L(U_q^r)$ if and only if $|y^*x| \leq H(y^*)$ for all $y^* \in (R^n)^*$ where $H(\cdot)$ is the support function of $L(U_q^r)$ [see 6]. Proceeding analogously to Conti [3], we obtain a necessary and sufficient condition for controllability with respect to the pair x_o,x_1.

THEOREM 1: (S) is controllable with respect to x_o,x_1 on $[t_o,T]$ using $U=U_q^r$ if and only if

$$|y^*(x_1-\phi(T,t_o)x_o)| \leq r\left(\sum_{i=1}^{k}\|y^*\phi(T,t_i)C_i\|_2^p\right)^{1/p} \text{ for}$$

all $y \in (R^n)^*$ where $1 < q \leq \infty$ and $1/p + 1/q = 1$.

If $U = U_1^r$ then the condition becomes

$$|y^*(x_1 - \phi(T,t_o)x_o)| \leq r \max_{1\leq i\leq k} \|y^*\phi(T,t_i)C_i\|_2.$$

In order to obtain global controllability results, we need to let the set of admissible controllers be $U=R^{mxk}$. In this instance global controllability is equivalent to the operator L being onto, i.e., $L(R^{mxk}) = R^n$. This latter condition is equivalent to the adjoint operator L^*, which takes $y \in R^n$ into

$$[y^*\phi(T,t_1)C_1,\ y^*\phi(T,t_2)C_2,\ldots,y^*\phi(T,t_k)C_k] \in R^{mxk},$$

being one-to-one. We therefore obtain the usual algebraic criterion for global controllability.

THEOREM 2: (S) is globally controllable on $[t_o,T]$ using $U=R^{mxk}$ if and only if:
$[\phi(T,t_1)C_1,\phi(T,t_2)C_2,\ldots,\phi(T,t_k)C_k]$ has full rank n.

We use theorem 2 to obtain an alternate analytical characterization of controllability which becomes useful when dealing with constrained controllability [2,4].

THEOREM 3: (S) is globally controllable on $[t_o,T]$ using $U=R^{mxk}$ if and only if

$$\min\left\{\sum_{i=1}^{k}\|y^*\phi(T,t_i)C_i\|_2 : \|y\|_2=1\right\} > 0,$$ equivalently

if and only if there exists a positive constant m_1 such that:

$$\sum_{i=1}^{k}\|y^*\phi(T,t_i)C_i\|_2 \geq m_1\|y\|_2 \text{ for all } y \in R^n.$$

Remark: By equivalence of norms in finite dimensional spaces, we obtain the following characterization: (S) is globally controllable on $[t_o,T]$ iff for each p, $1\leq p<\infty$ there exists a positive constant m_p such that

$$\left(\sum_{i=1}^{k}\|y^*\phi(T,t_i)C_i\|_2^p\right)^{1/p} \geq m_p\|y\|_2 \text{ for all } y \in R^n.$$

The case $p=\infty$ is treated similarly.

III. TIME-INVARIANT SYSTEMS

If we restrict ourselves to the case of constant systems, then the previous results for time-varying systems will assume simpler and more familiar forms. Furthermore, if we make the simplifying assumption that the B_i's commute with the coefficient matrix A then the jumps can in some sense be decoupled as the resulting form of the transition matrix suggests:

$$\phi(t,t_o)=\prod_{j=k}^{1}(I+B_j)\exp(A(t-t_o)).$$

Using standard arguments, we therefore obtain the following Kalman type controllability criterion [5].

THEOREM 4: Assume that:
(i) $\det(I+B_i)\neq 0 \quad i=1,\ldots,k$

(ii) A and B_i commute for all $i=1,\ldots,k$.

Then (S) is globally controllable with $U=R^{mxk}$ iff

$\mathrm{rank}[C_1,AC_1,\ldots,A^{n-1}C_1,C_2,AC_2,\ldots,A^{n-1}C_2,\ldots,C_k,AC_k,\ldots,$

$A^{n-1}C_k]$ is equal to n.

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