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Constrained Controllability of Linear Impulse Differential Systems

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ABSTRACT

We consider the following linear impulse differential control system

$$\begin{cases} \dot{x} = A(t)x & t \neq t_k \\ \Delta x = B_k x(t_k) + C_k u_k & t = t_k \end{cases}$$

where the control sequences $\{u_k\}$ belong to some set of admissible controllers that is restricted either by norm or by range. We then give a necessary and sufficient condition for global null controllability of time-varying systems and some sufficient conditions for global null controllability for time-invariant systems with special structures.

I. INTRODUCTION

Many dynamical systems are characterized by the fact that at certain moments in their evolution they undergo rapid changes. Most notably this occurs in certain biological systems, population systems and even in control systems such as in pulse frequency modulated control systems. In modeling such systems it is more tractable and convenient to neglect the duration of these rapid changes and assume the state changes by jumps. The mathematical models of such processes are so-called differential systems with impulse effect, i.e., a system of ordinary differential equations, together with relations defining the jump condition [1]. More specifically the model is given by:

$$\begin{cases} \dot{x} = f(x,t) & \text{when } h(x,t) \neq 0 \\ \Delta x = j(x,t) & \text{when } h(x,t) = 0 \end{cases}$$

where $t \in R$ is the time variable, $x \in R^n$ is the state vector, $f: R^n \times R \to R^n$ and $j: R^n \times R \to R^n$ defines the jump condition. A point (x,t) in the extended phase space follows the solution trajectory of the differential system and as soon as it hits the hypersurface σ of equation $h(x,t) = 0$, the system incurs an instantaneous jump of 'size' $j(x,t)$.

In this note, we deal solely with deterministic, linear impulse systems whose instants of impulse effect are fixed, i.e., represented by a sequence of time hyperplanes $t = t_k$ where $\{t_k\}$ is a given time sequence.

$$(S) \quad \begin{cases} \dot{x} = A(t)x & t \neq t_k \\ \Delta x = x(t_k^+) - x(t_k) = B_k x(t_k) & t = t_k \end{cases}$$

where

$$k \in N, \quad A(\cdot) \in PC(R_+, R^{n\times n}), \quad B_k \in R^{n\times n}$$

$$\lim_{x \to \infty} t_k = +\infty$$

If $\det(I + B_k) \neq 0$ for all $k \in N$ and if U_k denotes the transition matrix of $\dot{x} = A(t)x$ on $t_{k-1} < t < t_k$ then the transition matrix ϕ of (S) is

$$\phi(t,s) = U_k(t_k^+) \prod_{j=k}^{i+1} (I+B_j) U_j(t_j, t_{j-1}^+)(I+B_i) U_i(t_i, s)$$

Consider now the following control problem:

$$(S) \quad \begin{cases} \dot{x} = A(t)x & t \neq t_k \\ \Delta x = B_k x(t_k) + C_k u_k & t = t_k \end{cases}$$

where $C_k \in R^{n\times m}$ and $u_k \in U \subseteq R^m$ for $k \in N$ are the control vectors. The constrained null-controllability problem deals with the following question: Given an initial state $x(t_0) = x_0$ does there exist a sequence $\{u_k\}$ of admissible controllers that steers the system to the origin in a finite time T. In most treatment of constrained controllability the set of admissible controllers is restricted in various ways, either by norm or by range. In this note, we will give a necessary and sufficient condition for global null-controllability using controllers that are elements of unit balls of the sequence spaces ℓ_q^m (denoted by U_q). Furthermore, we provide some sufficient conditions for global null-controllability for systems with special properties.

II - RESULTS

We start this section by giving a general necessary and sufficient condition for global null-controllability. To motivate this basic criterion we introduce and briefly discuss a concept very similar to that of the reachable set. Consider the solution of system (S)

$$x(t,t_0,x_0) = \phi(t,t_0)x_0 + \sum_{t_0<t_i<t} \phi(t,t_i)C_i u_i$$

if we set $x(t, t_0, x_0) = 0$, we obtain using the nonsingularity of ϕ

$$x_0 = -\sum_{t_0<t_i<t} \phi(t_0,t_i)C_i u_i$$

We now let

$$R(t,t_0) = \left\{ x \in R^n : x = \sum_{t_0<t_i<t} \phi(t_0,t_i)C_i u_i \quad \text{for} \quad u_i \in U \right\}$$

clearly $R(t,t_0)$ consists of all initial positions $x_0 \in R^n$ that can be steered to the origin at or before time t. If there exists a time T such that $x_0 \in R(T, t_0)$ then system (S) is null-controllable for x_0. To achieve global null-controllability, a necessary and sufficient condition is

$$\bigcup_{t \geq t_0} R(t,t_0) = R^n .$$

This last observation will translate in a divergence condition for global null-controllability analogous to Conti's [2] for differential systems without impulses. We omit the proof as it uses similar basic arguments from convex analysis.

Theorem 2.1: Assume $\det(I + B_i) \neq 0$ for all $i \in N$. Then (S) is globally null-controllable by means of U_q if and only if

$$\lim_{t_i \to \infty} \sum_{t_0<t_i} |C_i^T \phi^T(t_0,t_i)y|_2^p = +\infty \qquad \text{for all}$$

nonzero $y \in R^n$ where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Furthermore (S) is globally null-controllable by means of U_1 if and only if

$$\lim_{k \to \infty} \max_{1 \leq i \leq k} |C_i^T \phi^T(t_0,t_i)y|_2 = +\infty$$

for all nonzero $y \in R^n$.

Theorem 2.1 constitutes a complete characterization of global null-controllability and clearly demonstrates the dependence of controllability on the transition matrix Φ, the control matrices C_i's and the set of admissible controllers through the exponent p. To insure divergence of the infinite series, basically three conditions have to be met:

1. The products $C_i^T \phi^T(\cdot,t_i)y$ cannot be identically zero for nonzero $y \in R^n$, in other words, for the system to be controllable with constrained controls it has to be controllable with unconstrained controls.
2. The matrices $\phi^T(\cdot,t_i)$ do not decay to zero too rapidly, i.e., the solutions of (S) do not grow too fast for the restricted controller to keep up.
3. The exponent p has to be the proper one and hence the appropriate set of admissible controllers has to be used.

We conclude from the above remarks that if a system is stable in the sense of bounded transition matrix (but not necessarily asymptotically or exponentially stable) and in some sense uniformly controllable with unconstrained controllers, we would expect it to be globally null-controllable with certain classes of admissible controllers (see Sontag and Sussman [5]). Indeed the next theorem shows that this is in fact true but before we state and prove the theorem, let us introduce the well known concept of uniform controllability by giving a formal definition, see Kalman [4] for more general definitions.

Definition 2.1: (S) is uniformly controllable on $[t_0, \infty]$ if there exist a positive integer r and a positive real number α such that for all positive integers $n \geq t_0$ we have

$$\sum_{i=n}^{r+n} \phi(t_{r+n},t_i)C_iC_i^T\phi^T(t_{r+n},t_i) \geq \alpha I$$

in the sense of quadratic form.

Theorem 2.2: Assume $\det(I + B_i) \neq 0$ for all $i \in N$. If (S) is uniformly controllable on $[t_0, \infty]$ and stable then it is globally null-controllable by means of U_q for all $1 < q \leq \infty$. Moreover if the system is asymptotically or exponentially stable it is also controllable by means of U_1.

Proof: To prove global null-controllability we use the divergence condition given in theorem 2.1. Consider the infinite series

$$\sum_{i=1}^{\infty} |C_i^T\phi^T(t_0,t_i)y|_2^p \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1 \tag{2.1}$$

(2.1) can be rewritten as

$$\sum_{n=0}^{\infty} \sum_{i=nr}^{(n+1)r} \left\{|C_i^T\phi^T(t_{(n+1)r},t_i)\phi^T(t_0,t_{(n+1)r})y|_2^2\right\}^{\frac{p}{2}}$$

(2.2)

Using the assumption of uniform controllability we obtain

$$\sum_{i=1}^{\infty} |C_i^T\phi^T(t_0,t_i)y|_2^p \geq \sum_{n=0}^{\infty} \alpha^{\frac{p}{2}} |\phi^T(t_0,t_{(n+1)r})y|_2^p \tag{2.3}$$

From the assumption of stability we know that

$|\phi(t,t_0)| \leq K$ for some $K > 0$ and all $t \geq t_0$,

therefore

$$|\phi^T(t_0,t_{(n+1)r})y|_2 \geq \frac{|y|_2^2}{|y^T\phi^T(t_{(n+1)r},t_0)|_2} \geq \frac{1}{K}|y|_2 \tag{2.4}$$

Therefore inequality (2.3) becomes

$$\sum_{i=1}^{\infty} |C_i^T\phi^T(t_0,t_i)y|_2^p \geq \sum_{n=0}^{\infty} \left(\frac{\sqrt{\alpha}}{K}\right)^p |y|_2^p \tag{2.5}$$

The right side of (2.5) clearly diverges, hence we have global null-controllability by means of all U_q, $1 < q \leq \infty$. A similar argument can be applied to show that if (S) is asymptotically or exponentially stable then it is globally null-controllable by means of U_q, $1 \leq q \leq \infty$.

In case (S) is time-invariant then we have the following corollary:

Corollary 2.1: If (S) is a stable, time-invariant system and

1) $\det(I + B_i) \neq 0$ for all $i \in N$
2) A and B_i commute for all $i \in N$
3) $\text{rank}\,[C_i, AC_i, \ldots, A^{n-1}C_i] = n$ for all $i \in N$.

Then (S) is globally null-controllable by means of U_q for all $1 < q \leq \infty$. Furthermore if (S) is asymptotically stable then it is globally null-controllable for all $1 \leq q \leq \infty$.

Remark 2.1: Note that stability and uniform controllability does not necessarily imply global null-controllability by means of U_1, i.e., the unit ball of ℓ_1^m. Indeed consider the easy example:

$$\dot{x} = 0 \qquad t \neq t_k$$

$$\Delta x = u_k \qquad t = t_k$$

It is clear the only initial conditions x_0 that can be steered to zero are such that $-1 \leq x_0 \leq 1$.

In the case of constant systems, if we impose some structural and growth conditions, we can apply theorem 2.1 to obtain various other criteria that are sufficient for global null-controllability. More explicitly suppose that the B_i's commute with the coefficient matrix A and the products $(I + B_i)\,\Psi(t_{i+1}, t_i)$ do not grow too fast, where $\Psi(t, t_0) = \exp(A(t-t_0))$, then we obtain the following sufficiency condition:

Theorem 2.3: Assume

i) $\det(I + B_i) \neq 0$ for all $i \in N$
ii) A and B_i commute for all $i \in N$
iii) $\text{rank}\,[C_i, AC_i, \ldots, A^{n-1}C_i] = n$ for all $i \in N$
iv) $\|(I + B_i)\|\,\|\Psi(t_{i+1}, t_i)\| \leq \gamma_i$ where γ_i's satisfy

$$\sum_{i=1}^{n} \ln\gamma_i = O\left(\ln n^{\frac{1}{p}}\right) \quad \text{as} \quad n \to \infty.$$

Then (S) is globally null-controllable by means of U_q for all q such that $1 < q \leq \infty$. (Note: $\frac{1}{p} + \frac{1}{q} = 1$)

Proof: We again rely on the criterion given in theorem 2.1 to show global null-controllability. Proceeding similarly as in the proof of theorem 2.2 we have:

$$\sum_{i=1}^{\infty} \|C_i^T \phi^T(t_0,t_i)y\|_2^p \geq \sum_{n=0}^{\infty} \alpha^{\frac{p}{2}} \|\phi^T(t_0,t_{(n+1)r})y\|_2^p .$$

Now

$$\|\phi(t_{(n+1)r},t_0)\| = \| \prod_{i=(n+1)r}^{1} (I+B_i) \prod_{i=0}^{(n+1)r-1} \Psi(t_{i+1},t_i)\| \leq \prod_{i=1}^{(n+1)r} \gamma_i \tag{2.6}$$

Since

$$\|\phi^T(t_0,t_{(n+1)r})y\|_2 \geq \frac{\|y\|_2}{\|y^T\phi^T(t_{n+1)r},t_0)\|}$$

(2.6) implies that

$$\sum_{i=1}^{\infty} \|C_i^T\phi^T(t_0,t_i)y\|_2^p \geq \sum_{n=0}^{\infty} (\sqrt{\alpha})^p \prod_{i=1}^{(n+1)r} \left(\frac{1}{\gamma_i}\right)^p \|y\|_2^p \tag{2.7}$$

assumption (iv) implies that

$$\prod_{i=1}^{(n+1)r} \left(\frac{1}{\gamma_i}\right)^p = 0\left(\frac{1}{n}\right)$$ therefore the right hand side of (2.7) diverges. This proves the theorem.

Finally we end this note by giving one more application of theorem 2.1 to a system with a special structure. Suppose system (S) is given by

$$\begin{cases} \dot{x} = Ax & t \neq t_k \\ \Delta x = \alpha_k x(t_k) + C_k u_k & t = t_k \end{cases}$$

then the transition matrix ϕ becomes

$$\phi(t,t_0) = \prod_{i=1}^{k} (1+\alpha_i) e^{A(t-t_0)} .$$

We therefore obtain the following sufficiency condition:

Theorem 2.4: Assume

i) $\alpha_i \neq -1$ for all $i \in N$

ii) rank $[C_i, AC_i, \ldots, A^{n-1}C_i] = n$ for all $i \in N$

iii) $Re(\lambda_i) \leq 0$ for all eigenvalues λ_i of A

iv) $\sum_{i=1}^{n} |\alpha_i| \leq \ln n^{\frac{1}{p}}$ for all $n \in N$

Then (S) is globally null-controllable by means of U_q for all q such that $1 < q \leq \infty$.

Proof: Proceeding similarly as before, we have

$$\sum_{i=1}^{\infty} \|C_i^T\phi^T(t_0,t_i)y\|_2^p \geq \sum_{n=0}^{\infty} \alpha^{\frac{p}{2}} \prod_{i=1}^{(n+1)r} (1+|\alpha_i|)^{-p} \|e^{-A^T(t_{(n+1)r}-t_0)}y\|_2^p .$$

Without loss of generality assume that $t_{(n+1)r} = t_0 + (n+1)r$. Since $(1+|\alpha_i|) \leq e^{|\alpha_i|}$ the previous inequality assumes the form:

$$\sum_{i=1}^{\infty} \|C_i^T\phi^T(t_0,t_i)y\|_2^p \geq \sum_{n=0}^{\infty} \alpha^{\frac{p}{2}} \exp\left(-p\sum_{i=1}^{(n+1)r} |\alpha_i|\right) \|\exp(-A^T(n+1)r)y\|_2^p$$

We explicitly bound $\|\exp(-A^T(n+1))ry\|_2^2$ from below, indeed for all nonzero $y \in R^n$

$$\|\exp(-A^T t)y\|_2^2 \geq \exp(-2\beta t)t^{2\nu}(a+a(t))$$

where $\beta \in R$, $\nu \in N$ and a(t) depend in general on the jordan canonical form of A and the vector y and satisfy

1. $\min_{1\leq i\leq s} Re(\lambda_i) \leq \beta \leq 0$

.2 $0 \leq \nu \leq \max_{1\leq i\leq s}(n_i - 1)$ and $\begin{cases} a(t) \equiv 0 & \text{if } \nu = 0 \\ a(t) \to 0 & \text{as } t \to \infty \text{ if } \nu > 0 \end{cases}$

.3 a>0 where $A \sim \oplus \sum_{i=1}^{s} J_i$ where J_i $i = 1,2,\cdots,s$

are jordon blocks of order n_i. Therefore

$$\sum_{i=1}^{\infty} \|C_i^T\phi^T(t_0,t_i)y\|_2^p \geq \sum_{n=0}^{\infty} \alpha^{\frac{p}{2}} \exp\left\{-p\left(\sum_{i=1}^{(n+1)r} |\alpha_i| + \beta(n+1)r\right)\right\} [(n+1)r]^{\nu p}(a+o(1))^{\frac{p}{2}} .$$

By assumptions (ii) and (iv) we obtain

$$\sum_{i=1}^{\infty} \|C_i^T\phi^T(t_0,t_i)y\|_2^p \geq \sum_{n=0}^{\infty} \alpha^{\frac{p}{2}} \frac{1}{(n+1)r} (a+o(1))^p$$

which is clearly a divergent series. This completes the proof of the theorem.

Remarks:

1. The assumption on the coefficient matrix A is that $Re(\lambda_i) \leq 0$ for all eigenvalues λ_i of A, therefore any repeated eigenvalue with zero real part gives rise to an unstable mode. So the theorem does take into consideration unstable systems (albeit polynomial growth instability).

2. Clearly in case (S) is asymptotically stable, i.e., $Re(\lambda_i) < 0$ for all eigenvalues λ_i of A then global null-controllability of (S) follows even if we used U_1 provided assumption (iv) is replaced

by $\sum_{i=1}^{n} |\alpha_i| < 2\beta(n+1)r$.

3. We can obtain a less conservative result if condition (iv) of the theorem is replaced by a condition that insures the divergence of the infinite product $(\Pi(1 + \alpha_i))^{-1}$.

4. If the jump matrix is constant, i.e., (S) has the form

$$\begin{cases} \dot{x} = Ax & t \neq t_k \\ \Delta x = Bx(t_k) + C_k u_k & t = t_k \end{cases}$$

and if A and B commute, then using similar arguments as above, it can be shown that (S) is globally null-controllable provided the Kalman rank condition holds and the moduli of the eigenvalues of $(I + B)\, e^A$ are less or equal to 1. (See [3])

III. CONCLUSION

In this note we gave a general necessary and sufficient condition for global null-controllability with constrained controls of differential systems with impulse effect. Relying on this criterion and the concept of uniform controllability in addition to certain growth conditions on the system transition matrix and the sizes of the jumps we obtain sufficiency conditions for global constrained controllability of certain stable and unstable systems.

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