Proceedings of the American Control Conference
San Diego, California • June 1999

Mixed $\mathcal{H}_2/\mathcal{L}_1$ Controllers for Continuous–Time MIMO Systems *

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Abstract

In this paper we consider the problem of optimizing the $\mathcal{H}_2$ norm, while keeping the $\mathcal{L}_1$ norms of some other transfer functions under specified levels. We show that the optimal closed-loop impulse responses of transfer functions in the constraints have finite support, and thus non-rational Laplace transforms. To solve the difficulty of implementing non-rational controllers, we propose a method for synthesizing rational controllers with performance arbitrarily close to optimal.

1 Introduction

Many control problems involve the optimization of certain performance measures, in addition to the stabilization of the system. Often minimization of a single performance index is not enough to capture several, perhaps conflicting design specifications, leading to a research effort aimed towards designing *multi–objective* feedback controllers, capable of satisfying multiple performance specifications (see for instance [4, 8] and references therein). In this paper we consider the problem of optimizing the $\mathcal{H}_2$ norm, subject to $\mathcal{L}_1$ constraints, leading to a mixed $\mathcal{H}_2/\mathcal{L}_1$ problem [1]. The discrete time version of the problem was studied in [9] in the SISO case, and [7] for MIMO systems (see also [6, 1], for the SISO $\ell_1/\mathcal{H}_2$ and $\mathcal{L}_1/\mathcal{H}_2$ problems). In this paper we explore the continuous–time counterpart of the problem. The main results of the paper show that the optimal solution has non–rational Laplace transform even if the original plant is rational, and propose a Euler Approximating System based rational controller synthesis method.

*This work was supported in part by NSF under grant ECS–9625920.

[1] This problem can also be motivated as an optimal $\mathcal{H}_2$ problem subject to robustness constraints.

2 Notation and System Preliminaries

The notations are standard. $\mathcal{H}_p$ ($\mathcal{H}_p^{m\times n}$), $\mathcal{L}_p$ ($\mathcal{L}_p^{m\times n}$), and ℓ_p ($\ell_p^{m\times n}$) are the standard notations for the commonly used Hardy and Banach spaces. AM denotes the space of all purely atomic measures on R_+, i.e., $AM = \{h, h(t) = \sum_{k=0}^{\infty} h_k\delta(t-t_k), \{h_k\} \in \ell_1\}$ with $\|h\|_{AM} \doteq \sum_{k=0}^{\infty}|h_k|$. A denotes the space whose elements have the form $h = h^L(t) + \sum_{k=0}^{\infty} h_k^l \delta(t-t_k)$ where $h^L(t) \in \mathcal{L}_1(R_+)$, $\{h_k^l\} \in \ell_1$, and $t_k \geq 0$ (*i.e.*, $A = AM \times \mathcal{L}_1(R_+)$), with $\|h\|_A \doteq \|h^L\|_{\mathcal{L}_1} + \|h^l\|_{\ell_1}$. $\widehat{x}(z)$ ($\widehat{x}(s)$) denotes the $\mathcal{Z}$ transform of a right sided real sequence $x = \{x(k)\}_{k=0}^{\infty}$ (the Laplace transform of a function $x(t)$ on R_+).

It is well known that the set of all achievable internally bounded–input bounded–output stable closed-loop maps is given by

$$\Theta = \{\Phi \in \mathcal{L}_1^{n_z\times n_w}(R_+) : \text{There exists } Q \in \mathcal{L}_1^{n_u\times n_y}(R_+) \text{ s.t. } \Phi = H - U * Q * V\}$$

where $H \in \mathcal{L}_1^{n_z\times n_w}(R_+)$, $U \in \mathcal{L}_1^{n_z\times n_u}(R_+)$, and $V \in \mathcal{L}_1^{n_y\times n_w}(R_+)$ are fixed maps that depend on the plant P, and $Q \in \mathcal{L}_1^{n_u\times n_y}(R_+)$ is a free parameter. In the sequel we assume, without loss of generality [3], that $\widehat{U}$ and $\widehat{V}$ have full column and row ranks respectively. Let the Smith-McMillan decomposition of $\widehat{U}$ and $\widehat{V}$ given by $\widehat{U} = \widehat{L}_U\widehat{M}_U\widehat{R}_U$ and $\widehat{V} = \widehat{L}_V\widehat{M}_V\widehat{R}_V$, where $L_U \in \mathcal{L}_1^{n_z\times n_z}(R_+)$, $R_U \in \mathcal{L}_1^{n_u\times n_u}(R_+)$, $L_V \in \mathcal{L}_1^{n_y\times n_y}(R_+)$, and $R_V \in \mathcal{L}_1^{n_w\times n_w}(R_+)$ are unimodular matrices. $M_U \in \mathcal{L}_1^{n_z\times n_u}(R_+)$ and $M_V \in \mathcal{L}_1^{n_y\times n_w}(R_+)$ can be written as

$$\widehat{M}_U = (diag\{\frac{\widehat{\epsilon}_1}{\widehat{\psi}_1}, \cdots, \frac{\widehat{\epsilon}_{n_u}}{\widehat{\psi}_{n_u}}\}\ 0_{n_u\times(n_z-n_u)})^T$$
$$\widehat{M}_V = (diag\{\frac{\widehat{\epsilon}'_1}{\widehat{\psi}'_1}, \cdots, \frac{\widehat{\epsilon}'_{n_y}}{\widehat{\psi}'_{n_y}}\}\ 0_{n_y\times(n_w-n_y)})$$

where $\{\widehat{\epsilon}_i, \widehat{\psi}_i\}$ and $\{\widehat{\epsilon}'_i, \widehat{\psi}'_i\}$ are coprime monic polynomial pairs. Let $\mathcal{S}_{UV}$ denote the set of zeros of $\widehat{U}$ and $\widehat{V}$ in the closed right half plane. We assume that neither $\widehat{U}$ nor $\widehat{V}$ have zeros on the $j\omega$

axis. For $s_o \in \mathcal{S}_{UV}$ define, $\sigma_{U_i}(s_o) \doteq$ multiplicity of s_o as a roots of $\widehat{\epsilon}_i$, $\sigma_{V_j}(s_o) \doteq$ multiplicity of s_o as a roots of $\widehat{\epsilon}'_j$. Also define the polynomial row and column vectors $\widehat{\alpha}_i$ and $\widehat{\beta}_j$ as $\widehat{\alpha}_i \doteq (\widehat{L}_U^{-1})_i$, $\widehat{\beta}_j \doteq (\widehat{R}_V^{-1})^j$, where $(M)_i$ and $(M)^j$ denote the ith row and jth column of the matrix M respectively. Further denote by α_{ip} and β_{jq} the pth column of α_i and qth row of β_j respectively, and define $F^{ijks_o} \in \mathcal{L}_\infty^{n_z \times n_w}(R_+)$ by,

$$F_{pq}^{ijks_o}(t) \doteq \int_0^\infty \int_0^\infty \alpha_{ip}(s-l)\beta_{jq}(l-t)(e^{-st})^{(k)} ds dl \Big|_{s=s_o}$$

for $1 \le p \le n_z$, $1 \le q \le n_w$, $k = 0, ..., \sigma_{U_i}(s_o) + \sigma_{V_j}(s_o) - 1$, $i = 1, ..., n_u$, and $j = 1, ..., n_y$, where $(.)^{(k)}$ denotes the kth derivative with respect to s. Finally, define $G_{\alpha_i qt} \in \mathcal{L}_1^{n_z \times n_w}(R_+)$ for $n_u + 1 \le i \le n_z$, $1 \le q \le n_w$ and $t \in R_+$, and $G_{\beta_j pt} \in \mathcal{L}_1^{n_z \times n_w}(R_+)$ for $n_y + 1 \le j \le n_w$, $1 \le p \le n_w$ and $t \in R_+$ by,

$$G_{\alpha_i qt}(l) \doteq (0_{n_z \times (q-1)} \; \alpha_i'(t-l) \; 0_{n_z \times (n_w - q)})$$
$$G_{\beta_j pt}(l) \doteq (0_{n_w \times (p-1)} \; \beta_j(t-l) \; 0_{n_w \times (n_z - p)})^T$$

Theorem 1 *[3] Define $RF^{ijks_o} := Real(F^{ijks_o})$ and $IF^{ijks_o} = Imaginary(F^{ijks_o})$ and assume that $S_{UV} \subset int(RHP)$. $\Phi \in \mathcal{L}_1^{n_z \times n_w}(R_+)$ is achievable if and only if the following conditions hold:*

$$\begin{array}{c} <\Phi, RF^{ijks_o}> = <H, RF^{ijks_o}> \\ <\Phi, IF^{ijks_o}> = <H, IF^{ijks_o}> \end{array} \quad (1)$$

for $s_o \in S_{UV}$, $i = 1, ..., n_u$, $j = 1, ..., n_y$, and $k = 0, ..., \sigma_{U_i}(s_o) + \sigma_{V_j}(s_o) - 1$, and

$$\begin{array}{c} <\Phi, G_{\alpha_i qt}> = <H, G_{\alpha_i qt}> \\ <\Phi, G_{\beta_j pt}> = <H, G_{\beta_j pt}> \end{array} \quad (2)$$

for $i = n_u + 1, ..., n_z$, $j = n_y + 1, ..., n_w$, $q = 1, ..., n_w$, $p = 1, ..., n_z$, and $t \in R+$.

3 The Mixed $\mathcal{H}_2/\mathcal{L}_1$ Control Problem

3.1 Problem Formulation

Define the following set of indices:

$N_w = \{1, \ldots, n_w\}$, $N_z = \{1, \ldots, n_z\}$

S: the subset of N_z corresponding to rows of Φ subject to an $\mathcal{L}_1$ constraint.

$\overline{M}$: set of indices (i, j) of transfer functions appearing in the $\mathcal{H}_2$ objective.

$\overline{N}$: set of indices (i, j) of transfer functions appearing in the $\mathcal{L}_1$ norm constraint.

$MN \doteq \overline{M} \cap \overline{N}$: functions appearing both in the objective and the constraints.

$M \doteq \overline{M} \backslash MN$: set of indices (i, j) such that the Φ_{ij} appears only in the objective function.

$N \doteq \overline{N} \backslash MN$: set of indices (i, j) such that Φ_{ij} appears only in the constraint.

Then the problem can be precisely stated as:

Problem 1 *Given the FDLTI plant P shown in Figure 1, find:*

$$\mu \doteq \inf_{\Phi \in \Gamma_\gamma} \{ \sum_{(p,q) \in \overline{M}} ||\Phi_{pq}||^2_{\mathcal{H}_2} \}$$

and the corresponding controller K, where

$$\Gamma_\gamma = \{\Phi \in \Theta : \textstyle\sum_{(p,q)\in\overline{M}} ||\Phi_{pq}||^2_{\mathcal{H}_2} < \infty, \; \sum_{q \in N_p} ||\Phi_{pq}||_A \le \gamma_p, \;\; \forall p \in S\}$$

where, for each $p \in S$, N_p denotes the elements of the p^{th} row subject to a constraint.

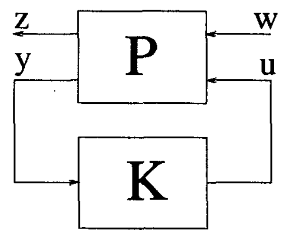

Figure 1: The generalized plant

We will assume that $n_z = n_u$ and that $n_y = n_w$, i.e., "one–block" category, where only the "zero interpolation" constraints (the first set of conditions in Theorem 1) are present [3]. However, the assumption can be relaxed to two and four–blocks via delay augmentation. We will further assume that for all $(p, q) \in N_z \times N_w$, the transfer function Φ_{pq} appears at least in the $\mathcal{L}_1$ constraint or in the objective function [2].

3.2 Primal and Dual Problems

Problem 2 (The Primal Problem)

$$\mu = \inf_{\Phi \in \Gamma_\gamma} \{\textstyle\sum_{(p,q)\in\overline{M}} ||\Phi_{pq}||^2_{\mathcal{H}_2}\}$$
$$s.t. \; <\Phi, F^{ijks_o}> = <H, F^{ijks_o}> \doteq b^{ijks_o}$$

for $s_o \in S_{UV}$, $i = 1, ..., n_u$, $j = 1, ..., n_y$, and $k = 0, ..., \sigma_{U_i}(s_o) + \sigma_{V_j}(s_o) - 1$.

[2] This can always be assumed without loss of generality by adding, if necessary, artificial constraints with arbitrarily high γ

Let $c_z \doteq \sum_{s_o \in S_{UV}} \sum_{i=1}^{n_u} \sum_{j=1}^{n_y} \sigma_{U_i}(s_o) + \sigma_{V_j}(s_o)$ and c_n denote the total number of zero interpolation and $\mathcal{L}_1$ constraints respectively. Define

$$\mathcal{A} \doteq \{\Phi^{n_z \times n_w} : \Phi_{pq} \in \mathcal{H}_2 \ \forall (p,q) \in M, \\ \Phi_{pq} \in A \ \forall (p,q) \in N, \Phi_{pq} \in \mathcal{H}_2 \cap \mathcal{L}_1 \ \forall (p,q) \in MN\}$$

Then, Lagrange's duality theorem [5] yields the following dual problem:

Problem 3 (The Dual Problem)

$$\mu = \max_{\bar{y} \in R^{c_n}, \bar{y} \geq 0, y \in R^{c_z}} \varphi(\bar{y}, y)$$

$$\varphi(\bar{y}, y) = \inf_{\Phi \in \mathcal{A}} \{\textstyle\sum_{(p,q) \in \overline{M}} \|\Phi_{pq}\|^2_{\mathcal{H}_2} \\ + \sum_{i,j,k,s_o} y_{ijks_o}(b^{ijks_o} - < F^{ijks_o}, \Phi >) \\ + \sum_{p \in S} \bar{y}_p (\sum_{q \in N_p} \|\Phi_{pq}\|_A - \gamma_p)\}$$

where $\bar{y}_p$ (an element of $\bar{y} \in R^{c_n}$) and y_{ijks_o} (an element of $y \in R^{c_z}$) are the Lagrange multipliers corresponding to $\mathcal{L}_1$ and zero interpolation constraints respectively.

Theorem 2 *If the solution $\Phi^o(t)$ exists, then*

$$\begin{aligned} \mu &= \max\{\textstyle\sum_{(p,q) \in M} \int_0^\infty -\Phi^2_{pq}(t)dt \\ &+ \textstyle\sum_{(p,q) \in MN} \int_0^\infty -\Phi^2_{pq}(t)dt \\ &+ \textstyle\sum_{i,j,k,s_o} y_{ijks_o} b^{ijks_o} - \sum_{p \in S} \bar{y}_p \gamma_p\} \end{aligned}$$

s.t. $\bar{y} \in R^{c_n}, \bar{y} \geq 0, y \in R^{c_z}, \Phi_{pq} \in \mathcal{L}_1(R_+) \cap \mathcal{H}_2(R_+) \ \forall (p,q) \in \overline{M}, \Phi_{pq} \in A \ \forall (p,q) \in N$, and

$$\begin{aligned} |Z_{pq}(t)| &\leq \bar{y}_p \ \text{if } (p,q) \in N \\ \Phi_{pq}(t) &= 0 \ \text{if } (p,q) \in N, |Z_{pq}(t)| < \bar{y}_p \\ 2\Phi_{pq}(t) &= Z_{pq}(t) - \bar{y}_p \ \text{if } (p,q) \in MN, Z_{pq}(t) > \bar{y}_p \\ &= Z_{pq}(t) + \bar{y}_p \ \text{if } (p,q) \in MN, Z_{pq}(t) < -\bar{y}_p \\ &= 0 \ \text{if } (p,q) \in MN, |Z_{pq}(t)| \leq \bar{y}_p \\ &= Z_{pq}(t) \ \text{if } (p,q) \in M \end{aligned}$$

$\forall t \in R_+$, where $Z_{pq}(t) \doteq \sum_{i,j,k,s_o} y_{ijks_o} F^{ijks_o}_{pq}(t)$. Furthermore, the optimal Φ^o_{pq} is unique $\forall (p,q) \in (MN) \cup M$.

3.3 Structure of the Optimal Solution

Lemma 1 *Assume that the $\mathcal{L}_1$ constraints are feasible. Then, for each p such that the corresponding $\mathcal{L}_1$ constraint is active, there exists $T \in R_+$ such that $\Phi^o_{pq}(t) = 0, \ \forall \ q, t \geq T$.*

Corollary 1 *Except in the trivial case where all the $\mathcal{L}_1$ constraints are inactive, the optimal $\mathcal{H}_2/\mathcal{L}_1$ closed loop transfer matrix and the optimal controller contain at least one element with a non–rational Laplace transform.*

4 Rational $\mathcal{H}_2/\mathcal{L}_1$ Controller Synthesis

From an engineering standpoint, given the difficulty of implementing non–rational transfer functions, this motivates the following problem:

Problem 4 (Rational $\mathcal{H}_2/\mathcal{L}_1$)

$$\mu^R \doteq \inf_{\Phi \in \mathcal{R}\Gamma_\gamma} \{ \sum_{(p,q) \in \overline{M}} \|\Phi_{pq}\|^2_{\mathcal{L}_2} \}$$

where $\mathcal{R}\Gamma_\gamma$ denotes the subspace of Γ_γ formed by functions having real rational Laplace transforms, and, given $\epsilon > 0$, a controller $\widehat{K}(s)$ such that the corresponding closed–loop transfer function $\Phi_R \in \mathcal{R}\Gamma_\gamma$ and satisfies $\sum_{(p,q) \in \overline{M}} \|\Phi_R\|^2_{\mathcal{L}_2} \leq \mu^R + \epsilon$.

Lemma 2 *For a given $\epsilon > 0$, there exist $\Phi_R \in \mathcal{R}\Gamma_\gamma$ such that $\left|\sum_{(p,q) \in \overline{M}} \|\Phi_R\|^2_{\mathcal{L}_2} - \mu\right| \leq \epsilon$.*

Corollary 2 $\mu = \mu^R$.

Given the continuous-time system:

$$\widehat{G}(s) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \qquad (3)$$

its EAS is defined as: [2, 1]:

$$\widehat{G}_E(z) = \left(\begin{array}{c|c} I + \tau A & \tau B \\ \hline C & D \end{array}\right) \qquad (4)$$

Theorem 3 *Consider a strictly decreasing sequence $\tau_i \to 0$. Define*

$$\begin{aligned} &\mu_i = \inf \tfrac{1}{\tau_i} \textstyle\sum_{(p,q) \in \overline{M}} \|\Phi_{E_{pq}}(k, \tau_i)\|^2_{\mathcal{H}_2} \\ &s.t. \textstyle\sum_{q \in N_p} \|\Phi_{E_{pq}}(k, \tau_i)\|_{\ell_1} \leq \gamma_p, \forall p \in S \end{aligned} \qquad (5)$$

Assume that $\gamma_p^{o,1} < \gamma_p, \forall p \in S$. Then, the sequence μ_i is non increasing and $\mu_i \to \mu^R$.

Remark 1 *From Theorem 3 it follows that the EAS based method can be used to solve problem 4, provided that the corresponding discrete–time problem can be solved.*

In the sequel, we show that these problems can be solved by using the algorithm proposed in [7], provided that it is *appropriately* modified so that the resulting closed–loop system is strictly proper. This guarantees that the corresponding continuous–time system has a finite $\mathcal{H}_2$ norm.

Consider the $\mathcal{H}_2/\ell_1$ problem for the EAS system. All internally stable closed-loop maps are given by $\Phi_E = H - U_E * Q_E * V_E$, where $H_E \in \ell_1^{n_z \times n_w}$, $U_E \in \ell_1^{n_z \times n_u}$, and $V_E \in \ell_1^{n_y \times n_w}$ are the EAS of

H, U, and V respectively, and $Q_E \in \ell_1^{n_u \times n_y}(R_+)$ is a free parameter. The $\mathcal{H}_2/\ell_1$ problem for the EAS system is given by,

$$\begin{array}{l} \mu_E = \inf_{\Phi_E \in \Gamma_\gamma}\{\sum_{(p,q)\in \overline{M}} \|\Phi_{Epq}\|_{\ell_2}^2\} \\ \text{subject to } < \Phi_E, F_E^{ijk\lambda_o} >= b_E^{ijk\lambda_o} \end{array} \quad (6)$$

$\lambda_o \in \Lambda_{UV}, i = 1, ..., n_u, j = 1, ..., n_y, k = 0, ..., \sigma_{U_i}(\lambda_o) + \sigma_{V_j}(\lambda_o) - 1$. In order to have finite $\mathcal{H}_2$ norm, $\widehat{\Phi}_{Epq}$ must be strictly proper for all $(p,q) \in \overline{M}$, or $\widehat{\Phi}_{Epq}(\infty) = \Phi_{Epq}(0) = 0 \ \forall (p,q) \in \overline{M}$. This results in the following problem,

$$\begin{array}{l} \mu_E = \inf_{\Phi_E \in \Gamma_\gamma}\{\sum_{(p,q)\in \overline{M}} \|\Phi_{Epq}\|_{\ell_2}^2\} \\ \text{subject to } < \Phi_E, F_E^{ijk\lambda_o} >= b_E^{ijk\lambda_o} \\ \text{and } \Phi_{Epq}(0) = 0 \ \forall (p,q) \in \overline{M} \end{array} \quad (7)$$

Note each element of $\widehat{\Phi}$ is given by:

$$\widehat{\Phi}_{Epq} = \widehat{H}_{Epq} - \sum_{m=1}^{n_u}\sum_{n=1}^{n_y} \widehat{U}_{Epm}\widehat{Q}_{Emn}\widehat{V}_{Enq}$$

In the case where $\widehat{H}_{Epq}$ and at least either $\widehat{U}_{Epm}$ or $\widehat{V}_{Enq}$ are strictly proper for all pairs (m,n) and $(p,q) \in \overline{M}$, the additional condition is automatically satisfied, and (7) is equivalent to,

$$\mu_E = \inf_{\Phi \in \Gamma_\gamma} \sum_{(p,q)\in \overline{M}} \|S_L * \Phi_{Epq}\|_{\ell_2}^2 \quad (8)$$

subject to

$$\sum_{q\in N_p,(p,q)\in N} \|\Phi_{Epq}\|_{\ell_1} + \sum_{q\in N_p,(p,q)\in MN} \|S_L * \Phi_{Epq}\|_{\ell_1} \le \gamma_p \ \forall p \in S$$

$$\sum_{(p,q)\in N} < \Phi_{Epq}, F_{Epq}^{ijk\lambda_o} > + \sum_{(p,q)\in \overline{M}} < S_L * \Phi_{Epq}, S_L * F_{Epq}^{ijk\lambda_o} >= b_E^{ijk\lambda_o}$$

where S_L denotes the left shift operator. After finding the optimal solution for this problem, one can shift it back to obtain the optimal Φ_E^o.

Consider now the case where either $\widehat{H}_{Epq}$ is proper but not strictly proper for some $(p,q) \in \overline{M}$, or the product $\widehat{U}_{Epm}\widehat{V}_{Enq}$ is proper (not strictly proper) for some m, n, $(p,q) \in \overline{M}$. Denote the set of indices (p,q) of $\Phi_{Epq} \in \overline{M}$ which has $\widehat{H}_{Epq}$ being proper or $\widehat{U}_{Epm}\widehat{V}_{Enq}$ being proper for some m,n such that $1 \le m \le n_u$ and $1 \le n \le n_y$, by $P \subset \overline{M}$. Define

$$\overline{\widehat{H}}_E \doteq col\{\widehat{H}_{Epq}\} \ (p,q) \in P$$

For $i = 1, ..., n_u$, define

$$\begin{array}{l} \overline{\widehat{Q}}_{Ei} \doteq [\widehat{Q}_{Ei1} \ \widehat{Q}_{Ei2} \ \cdots \ \widehat{Q}_{Ein_y}] \\ \overline{\widehat{Q}}_E \doteq [\overline{\widehat{Q}}_{E1}, \cdots, \overline{\widehat{Q}}_{En_u}]' \end{array}$$

Also for $i = 1, \cdots, n_u$, define

$$\begin{array}{l} \overline{\widehat{UV}}_{Ei} \doteq [\widehat{U}_{Epi}\widehat{V}_{E1q} \ \widehat{U}_{Epi}\widehat{V}_{E2q} \ \cdots \ \widehat{U}_{Epi}\widehat{V}_{En_yq}] \\ \overline{\widehat{UV}}_E \doteq col\{[\overline{\widehat{UV}}_{E1}\overline{\widehat{UV}}_{E2} \ \cdots \ \overline{\widehat{UV}}_{En_u}]\} \end{array}$$

for $(p,q) \in P$. Note that $\overline{\widehat{H}}_E \in card(P) \times 1$, $\overline{\widehat{Q}}_E \in (n_u \cdot n_y) \times 1$, and $\overline{\widehat{UV}}_E \in card(P) \times (n_u \cdot n_y)$, where $card(P)$ is the number of elements in P. Clearly, for the $\mathcal{H}_2/\mathcal{L}_1$ problem to have a finite solution, we must have

$$\begin{array}{l} \overline{\widehat{UV}}_E(\infty)\overline{\widehat{Q}}_E(\infty) = \overline{\widehat{H}}_E(\infty) \\ \Leftrightarrow \overline{UV}_E(0)\overline{Q}_E(0) = \overline{H}_E(0) \end{array} \quad (9)$$

The solution $\overline{Q}_E(0)$ to this problem is not necessarily unique since the number of row of $\overline{UV}_E$ is less than or equal to the number of column of $\overline{UV}_E$. Define

$$\widetilde{\widehat{H}}_{Epq} \doteq \widehat{H}_{Epq} - \sum_{m=1}^{n_u}\sum_{n=1}^{n_y} \widehat{U}_{Epm}\overline{\overline{Q}}_{Emn}(0)\widehat{V}_{Enq}$$

$\forall \ (p,q) \in P$ where $\overline{\overline{Q}}_{Emn}(0)$ is a matrix whose elements are constructed back from $\overline{Q}_E(0)$ in (9). Furthermore, define

$$\begin{array}{l} \widetilde{\widehat{U}}_{Epq} = \frac{\widehat{U}_{Epq}}{z}, \ \forall (p,q) \in P \\ \widetilde{\widehat{Q}}_E(z) = z(\widehat{Q}_E(z) - \overline{\overline{Q}}_E(0)) \end{array}$$

Then our problem can be written as,

$$\begin{array}{l} \mu_E = \inf_{\Phi_E \in \Gamma_\gamma} \|\Phi_{Epq}\|_{\ell_2}^2 \\ \text{s.t. } \sum_{q\in N_p} \|\Phi_{Epq}\|_{\ell_1} \le \gamma_p \ \forall p \in S \\ \text{and } < \Phi_E, \widetilde{F}_E^{ijk\lambda_o} >= \widetilde{b}_E^{ijk\lambda_o} \end{array} \quad (10)$$

where $\widetilde{F}_E^{ijk\lambda_o}$ and $\widetilde{b}_E^{ijk\lambda_o}$ are the zero interpolation condition for $\widetilde{H}_E, \widetilde{U}_E$ and V_E, where

$$\begin{array}{l} \widetilde{U}_E \doteq \\ \{\widetilde{U}_{Epm} \ \forall p \text{ s.t. } (p,q) \in P \text{ and } \forall m, U_{Epm} \text{ elsewhere}\} \\ \widetilde{H}_E \doteq \{\widetilde{H}_{Epq} \ \forall (p,q) \in P, H_{Epq} \text{ elsewhere}\} \end{array}$$

Since $\widetilde{\widehat{H}}_{Epq}$ and either $\widetilde{\widehat{U}}_{Epm}$ or $\widetilde{\widehat{V}}_{Enq}$ are both strictly proper for all m, n, and $(p,q) \in P$, $\widehat{\Phi}_{pq}$ is strictly proper for all $(p,q) \in \overline{M}$. Thus (10) is equivalent to:

$$\mu_E = \inf_{\Phi \in \Gamma_\gamma}\{\sum_{(p,q)\in \overline{M}} \|S_L * \Phi_{pq}\|_{\ell_2}^2\} \quad (11)$$

subject to

$$\sum_{q\in N_p,(p,q)\in N} \|\Phi_{Epq}\|_{\ell_1} + \sum_{q\in N_p,(p,q)\in MN} \|S_L * \Phi_{pq}\|_{\ell_1} \le \gamma_p \ \forall p \in S$$

$\sum_{(p,q)\in N} < \Phi_{Epq}, \widetilde{F}_{Epq}^{ijk\lambda_o} >$
$+ \sum_{(p,q)\in \overline{M}} < S_L * \Phi_{Epq}, S_L * \widetilde{F}_{Epq}^{ijk\lambda_o} >= \widetilde{b}_E^{ijk\lambda_o}$

The optimal $\Phi_E^o(k)$ can be recovered by shifting back the solution obtained from the problem (11).

5 An Example

Consider the following realization:

$$\begin{pmatrix} z_1 \\ z_2 \\ y \end{pmatrix} \doteq \begin{pmatrix} u_1 \\ u_2 \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & \frac{1}{s-1} & 1 \end{pmatrix} \begin{pmatrix} w \\ u_1 \\ u_2 \end{pmatrix}$$

Suppose that we want to minimize the $\mathcal{H}_2$ norm of T_{zw} subject to the constraint $\|T_{zw}\|_{\mathcal{L}_1} \leq 5$. With $\tau = 0.1$, and using the method in [7], the problem was reduced to a finite dimensional convex optimization problem. The solution was obtained with the optimal Φ_{11} of 40 th order ($\Phi_{21} = 0$). The optimal cost and $\mathcal{L}_1$ norm for different values of τ are given in Table 1. It can be seen that the smaller value of τ gives better cost. Finally, after the model reduction, the order of the controller was reduced to 8 with less than 1 percent performance loss. The reduced order controller is given by:

$$\widehat{K}(s) = \begin{pmatrix} \frac{-4.8535(s+9.0845)(s+0.1792)(s+0.0124)}{(s+9.0845)(s+3.1762)(s+0.1792)(s+0.0124)} \times \\ \frac{(s^2+4.8960s+30.0843)(s^2+1.7206s+4.5698)}{(s^2+6.1900s+33.8877)(s^2+1.6601s+4.2475)} \\ 0 \end{pmatrix}$$

Table 1: Cost for different τ

τ	$\|\Phi\|_{\mathcal{L}_1}$	$\|\Phi\|_{\mathcal{H}_2}$
$\tau = 0.05$	4.9883	2.8745
$\tau = 0.10$	4.9298	2.8886
$\tau = 0.15$	4.8589	2.9107
$\tau = 0.20$	4.7965	2.9348
Unconstrained $\mathcal{L}_1$	3.6	∞
Unconstrained $\mathcal{H}_2$	5.7389	2.8280

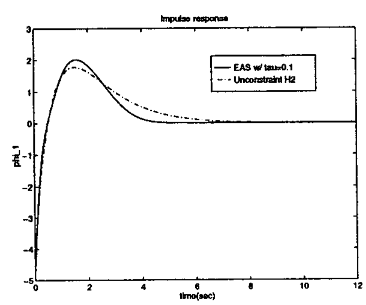

Figure 2: Impulse responses for $\tau = 0.1$

6 Conclusions and Further Research

In this paper we consider the continuous–time counterpart of the mixed $\mathcal{H}_2/\ell_1$ problem introduced in [7]. We first show that the continuous–time mixed $\mathcal{H}_2/\mathcal{L}_1$ problem leads to solutions having non–rational transfer functions, even when the original plant is rational.

Given the difficulties entailed in physically implementing a non–rational controller, in the second part of the paper we explore the restriction of the problem to rational functions. We show that the optimal cost can be approximated arbitrarily close by rational controllers that can be be synthesized by solving an auxiliary discrete–time non–standard $\mathcal{H}_2/\ell_1$ problem.

The systems considered in this paper are one-block. However, the technique can be extend to two and four–blocks via delay augmentation (a similar technique is proposed in [7] to handle 2 and 4 blocks discrete–time $\mathcal{H}_2/\ell_1$ problems).

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