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Mixed $\mathcal{L}_1/\mathcal{H}_2$ Controllers for Continuous Time Systems [1]

Takeshi Amishima, Juanyu Bu, Mario Sznaier
Department of Electrical Engineering
The Pennsylvania State University
University Park, PA 16802
email {msznaier, takeshi, jbu}@strider.ee.psu.edu

Abstract

In this paper we consider the problem of minimizing the $\mathcal{L}_1$ norm of a closed-loop transfer function while keeping its $\mathcal{H}_2$ norm under a specified level. It will be shown that the optimal closed-loop impulse response has finite support, and thus a non-rational Laplace transform. To solve this difficulty we propose a method for synthesizing rational controllers with performance arbitrarily close to optimal.

1 Introduction

Many control problems involve the minimization of certain performance measures, in addition to the stabilization of the system. Often optimization of a single performance index is not enough to capture several, perhaps conflicting design specifications, making the imposition of additional constraints necessary. This leads to the design of feedback controllers satisfying mixed performance specifications. Many types of mixed objective control problem have been studied recently, depending on how the exogenous inputs and regulated outputs are modeled and on how the additional constraints are imposed (see for instance [8, 12] and references therein). It is well known that the case where the exogenous inputs and outputs are measured in terms of peak time-domain values leads to an $\mathcal{L}_1$ optimization problem. If, in addition, one wishes to guarantee a certain level of nominal $\mathcal{H}_2$ performance, the problem becomes a mixed $\mathcal{L}_1/\mathcal{H}_2$ problem. The discrete time version of the problem was formulated and solved in [10],[11] (see also [15] for the related $\mathcal{H}_2/l_1$ problem). In this paper we explore its continuous-time counterpart. The main results of the paper show that the optimal solution has a non-rational Laplace transform, even if the plant and all weights are rational and proposes a technique to obtain ϵ-suboptimal rational approximations.

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2 Preliminaries

2.1 Notation

l_1 denotes the Banach space of right-sided, absolutely summable real sequence $x = \{x(k)\}_{k=0}^{\infty}$ equipped with the norm $||x||_{l_1} \doteq \sum_{k=0}^{\infty} |x(k)| < \infty$. l_2 denote the Hilbert space of energy bounded real sequences $x = \{x(k)\}_{k=0}^{\infty}$ equipped with the norm $||x||_{l_2} \doteq (\sum_{k=0}^{\infty} |x(k)|^2)^{\frac{1}{2}} < \infty$. R_+ denotes the set of nonnegative real numbers. $\mathcal{L}_1(R_+)$ denotes the Banach space of Lebesgue integrable functions $x(t)$ on R_+ equipped with the norm $||x||_{\mathcal{L}_1} \doteq \int_0^\infty |x(t)|dt < \infty$. $\mathcal{L}_2(R_+)$ denotes the normed space of Lebesgue integrable functions $x(t)$ on R_+ equipped with the norm $||x||_{\mathcal{L}_2} \doteq (\int_0^\infty |x(t)|^2 dt)^{\frac{1}{2}} < \infty$. $\mathcal{H}_2$ denotes the isometrically isomorphic space of $\mathcal{L}_2(R_+)$ (or l_2) under the Laplace transform $X(s)$ (or the $\mathcal{Z}$ transform $X(z)$) with norm given by $||X||_{\mathcal{H}_2} = ||x||_{\mathcal{L}_2}$ (or $||X||_{\mathcal{H}_2} = ||x||_{l_2}$). In the sequel we will use lower case letters to denote time-domain functions and upper-case letter to denote their Laplace (or $\mathcal{Z}$) transform, unless it is specifically indicated otherwise. Finally, the prefix $\mathcal{R}$ will be used to denote subspaces formed by functions having real rational Laplace (or $\mathcal{Z}$) transforms.

2.2 Lagrange duality

Theorem 1 (Lagrange duality, [9], [15],[10]) *Let X be a Banach space, Ω be a convex subset of X, Y be a finite dimensional space, Z be a normed space with positive cone P. Let $f(x)$ be a real valued convex functional, $G(x)$ be a convex mapping of X into Z, and $H(x)$ an affine linear* *map of X into Y such that $0 \in int[range(H)]$. Define*

$$\mu = \inf\{f(x) : G(x) \le 0, H(x) = 0, x \in \Omega\} \qquad (1)$$

Suppose that there exists $x \in \Omega$ such that $G(x) < 0$ and $H(x) = 0$ and suppose μ is finite. Then, $\mu = \max_{y_1 \ge 0, y_1 \in Z^*, y_2 \in Y} \varphi(y_1, y_2)$, where $\varphi(y_1, y_2) = \inf_{x \in \Omega}[f(x) + < G(x), y_1 > + < H(x), y_2 >]$ and the maximum is achieved for some $y_1^o \ge 0$, $y_2^o \in Z^*$ and $y_2^o \in Y$. Furthermore, if the infimum in (1) is achieved by some $x^o \in \Omega$, then $< G(x^o), y_1^o > + < H(x^o), y_2^o >= 0$ and $x^o \in \Omega$ minimizes $f(x) + < G(x), y_1^o > + < H(x), y_2^o >$, $x \in \Omega$.

3 The mixed $\mathcal{L}_1/\mathcal{H}_2$ control problem

In this section we formulate the mixed $\mathcal{L}_1/\mathcal{H}_2$ control problem, and show that the optimal closed loop has an irrational Laplace transform.

3.1 Problem setting

Problem 1 *Given the continuous–time FDLTI plant P shown in Figure 1 find:*

$$\mu^o = \inf_{K(s):stabilizing, \phi(t) \in S} \|\phi\|_{\mathcal{L}_1}$$

subject to $\|\phi\|^2_{\mathcal{L}_2} \le \gamma^2$ where $S = \mathcal{L}_1(R_+) \cap \mathcal{L}_2(R_+)$, and the corresponding controller $K(s)$.

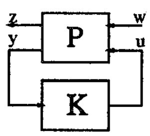

Figure 1: The generalized plant

In the sequel we will solve this problem by exploiting the Lagrange Duality Theorem[1].

3.2 The Primal and dual problems

It is well known that the set of all achievable internally stable closed-loop maps is given by

$$\phi(t) = t_1(t) - t_2(t) * q(t) \qquad (2)$$

where $t_1(t), t_2(t) \in \mathcal{L}_1(R_+)$ are fixed SISO maps and $q(t) \in \mathcal{L}_1(R_+)$ is a free parameter. To simplify our problem, in the sequel we assume that the map $T_2(s)$ has no zeros on the jw axis and all of its non–minimum phase zeros $s_1, \cdots, s_n$ are simple and real. Under these assumptions a given $\phi(t)$ is a feasible closed-loop map if and only if the interpolation conditions $\Phi(s_i) = T_1(s_i),\ i = 1, \cdots, n$ are satisfied. Equivalently, $\int_0^\infty A(t)\phi(t)dt = b$ where

$$A(t) = \begin{pmatrix} e^{-s_1 t} \\ \vdots \\ e^{-s_n t} \end{pmatrix},\ b = \begin{pmatrix} T_1(s_1) \\ \vdots \\ T_1(s_n) \end{pmatrix} \qquad (3)$$

Thus the primal problem can be stated as:

Problem 2 (Primal) *Find*

$$\mu^o = \inf_{\phi(t) \in S} \|\phi\|_{\mathcal{L}_1}$$

subject to $\|\phi\|^2_{\mathcal{L}_2} \le \gamma^2$, $\int_0^\infty A(t)\phi(t)dt = b$

Setting $X = \Omega = S$, $Y = R^n$, and $Z = R$ leads to the following dual problem:

Problem 3 (Dual) *Find*

$$\mu^o = \max_{y_1 \ge 0, y_1 \in R, y_2 \in R^n} \varphi(y_1, y_2)$$

where $\varphi(y_1, y_2) = \inf_{v(t) \in S}[\|v\|_{\mathcal{L}_1} + < \|v\|^2_{\mathcal{L}_2} - \gamma^2, y_1 > + < b - \int_0^\infty A(t)v(t)dt, y_2 >]$

Lemma 1 *Problem 3 is equivalent to:*

$$\mu^o = \max_{y_1 > 0, y_1 \in R, y_2 \in R^n, v \in S} -\|v\|^2_{\mathcal{L}_2} y_1 - y_1\gamma^2 + b^* y_2$$

subject to

$$v(t) = \begin{cases} \frac{-1-z(t)}{2y_1} & if \quad z(t) < -1 \\ \frac{1-z(t)}{2y_1} & if \quad z(t) > 1 \\ 0 & if \quad |z(t)| \le 1 \end{cases} \qquad (4)$$

where $z(t) = -A^* y_2$.

Remark 1 *Note that y_1 is always strictly greater than 0. This is due to the fact that the unconstrained optimal $\mathcal{L}_1$ solution (corresponding to $y_1 = 0$) is not in $\mathcal{H}_2$ (except in the trivial case). Thus the $\mathcal{H}_2$ constraint is always active for $\gamma^2 \in [\gamma_2, \infty)$, where*

$$\gamma_2 = \inf_{v \in S} \|v\|^2_{\mathcal{L}_2} \text{ subject to } \int_0^\infty A(t)v(t)dt = b$$

leading to optimal solutions essentially different from those in [6].

[1] For simplicity we consider SISO systems. However, the results here can be generalized to the MIMO case using the techniques in [11].

3.3 Structure of the optimal solution

Here we show that the optimal closed-loop impulse response $\phi^o(t)$ has finite support, and thus a non-rational Laplace transform $\Phi^o(s)$.

Corollary 1 *If the solution to the primal problem $\phi^o(t)$ exists, then $\phi^o(t) = v^o(t)$.*

Corollary 2 *If the $\mathcal{H}_2$ constraint is feasible (i.e., $\gamma^2 \in [\gamma_2, \infty)$), then there exists $T \in R_+$ such that $v^o(t) = 0$, $\forall t \geq T$.*

Corollary 3 *$\Phi^o(s)$ is irrational.*

4 Rational controller synthesis

So far we have shown that the solution to the primal problem (if it exists) is non-rational. From an engineering standpoint, given the difficulty of implementing non-rational transfer functions, this motivates the following problem:

Problem 4 (Rational $\mathcal{L}_1/\mathcal{H}_2$) *Find:*

$$\mu^R = \inf_{K(s):stabilizing, \phi(t) \in \mathcal{RS}} ||\phi||_{\mathcal{L}_1}$$

subject to $||\phi||^2_{\mathcal{L}_2} \leq \gamma^2$ where $\mathcal{RS} = \mathcal{RL}_1(R_+) \cap \mathcal{RL}_2(R_+)$, and, given $\epsilon > 0$, a controller $K(s)$ such that the corresponding closed-loop ϕ_R satisfies $||\phi_R||^2_{\mathcal{H}_2} \leq \gamma^2$ and $||\phi_R||_{\mathcal{L}_1} \leq \mu^R + \epsilon$.

In this section we derive a solution to Problem 4, based upon solving a modified auxiliary discrete-time $l_1/\mathcal{H}_2$ problem obtained using the Euler Approximating System (EAS). Recall that given the continuous-time system:

$$G(s) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{5}$$

its EAS is defined as the following discrete-time system [2]:

$$G_E(z) = \left(\begin{array}{c|c} I + \tau A & \tau B \\ \hline C & D \end{array}\right) \tag{6}$$

It is easily seen that we can obtain the EAS of $G(s)$ by the simple transformation $s = \frac{z-1}{\tau}$, i.e., $G_E(z) = G(\frac{z-1}{\tau})$.

Theorem 2 *Consider the stable strictly proper system $G(s) = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right)$ and its corresponding EAS $G_E(z,\tau) = \left(\begin{array}{c|c} I + \tau A & \tau B \\ \hline C & 0 \end{array}\right)$ where $\tau > 0$. Let $\tau_{max} = \min_{\lambda \in \Lambda} 2\frac{-re(\lambda)}{|\lambda|^2}$ where Λ is the set of eigenvalues of A and consider a strictly decreasing sequence $\tau_{max} > \tau_i \downarrow 0$. Then the following properties hold:*

1. *$G_E(z, \tau_i)$ is asymptotically stable for all i.*
2. *$||G||^2_{\mathcal{H}_2} \leq \frac{1}{\tau_i}||G_E(z,\tau_i)||^2_{\mathcal{H}_2}, \forall i$*
3. *$\frac{1}{\tau_i}||G_E(z,\tau_i)||^2_{\mathcal{H}_2} \geq \frac{1}{\tau_j}||G_E(z,\tau_j)||^2_{\mathcal{H}_2}, i < j$*
4. *$\lim_{\tau_i \to 0} \frac{1}{\tau_i}||G_E(z,\tau_i)||^2_{\mathcal{H}_2} = ||G||^2_{\mathcal{H}_2}$*

Next we state the main result of this section showing that Problem 4 can be solved by solving a sequence of discrete-time mixed $l_1/\mathcal{H}_2$ problems.

Theorem 3 *Consider a strictly decreasing sequence $\tau_i \to 0$. Define*

$$\mu_i = \min_{||g_E(k,\tau_i)||^2_{l_2} \leq \tau_i \gamma^2} ||g_E(k,\tau_i)||_{l_1}. \tag{7}$$

Assume that $\gamma_2 < \gamma^2$. Then, the sequence μ_i is non increasing and such that $\mu_i \to \mu^R$.

We show now that these problems can be solved by using the algorithm proposed in [10], provided that it is *appropriately* modified so that the resulting closed-loop system is strictly proper.

Consider the $l_1/\mathcal{H}_2$ problem for the EAS system. All internally stable closed-loop maps are given by

$$\phi_E(k) = t_{1E}(k) - t_{2E}(k) * q_E(k) \tag{8}$$

where $t_{1E}(k) \in l_1$ and $t_{2E}(k) \in l_1$ are the EAS of $t_1(t)$ and $t_2(t)$ respectively, and $q_E(k) \in l_1$ is a free SISO parameter. Let s_i denote the zeros of $T_2(s)$ in the open right half plane and define $z_i = 1 + \tau s_i$, $i = 1, ..., n$. Note that z_i are precisely the non-minimum phase zeros of $T_{2E}(z)$. Then $\phi(k)$ is a feasible closed map if and only if the interpolation condition $\Phi_E(z_i) = T_{1E}(z_i)$ for $i = 1, ..., n$ is satisfied. Specifically, $A_E \phi_E = b_E$ where:

$$A_E = \begin{pmatrix} 1 & \frac{1}{z_1} & \frac{1}{z_1^2} & \cdots \\ 1 & \frac{1}{z_2} & \frac{1}{z_2^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{1}{z_n} & \frac{1}{z_n^2} & \cdots \end{pmatrix}, \; b_E = \begin{pmatrix} T_{1E}(z_1) \\ T_{1E}(z_2) \\ \vdots \\ T_{1E}(z_n) \end{pmatrix} \tag{9}$$

The $l_1/\mathcal{H}_2$ problem for the EAS system is given by,

$$\mu_B = \inf_{q \in l_1} ||t_{1B} - t_{2B} * q_B||_{l_1}$$

subject to

$$||t_{1B} - t_{2B} * q_B||^2_{l_2} \leq {\gamma_B}^2, A_B \phi_B = b_B \tag{10}$$

In order for Theorem 2 to be valid, we must add an additional constraint to the optimization problem (10). Namely, $\Phi_E(z) = T_{1E}(z) - T_{2E}Q_E(z)$ must be strictly proper, or $\Phi_E(\infty) = \phi_E(0) = 0$.

This results in the following nonstandard $l_1/\mathcal{H}_2$ optimization problem,

$$\mu_B = \inf_{q \in l_1} ||t_{1B} - t_{2B} * q_B||_{l_1}$$

subject to

$$||t_{B1} - t_{B2} * q_B||_{l_2}^2 \leq \gamma_B{}^2, A_B\phi_B = b_B, \phi_B(0) = 0 \quad (11)$$

In the case when $T_{1E}(z)$ and $T_{2E}(z)$ are both strictly proper, the additional condition is automatically satisfied, and (11) is equivalent to,

$$\mu_B = \inf_{q \in l_1} ||S_L * (t_{1B} - t_{2B} * q_B)||_{l_1}$$

subject to

$$\begin{array}{l} ||S_L * (t_{E1} - t_{E2} * q_E)||_{l_2}^2 \leq \gamma_E{}^2 \\ A_E(S_L * \phi_E) = \bar{b}_E \\ \bar{b}_E = \begin{pmatrix} z_1 T_{1E}(z_1) \\ z_2 T_{1E}(z_2) \\ \vdots \\ z_n T_{1E}(z_n) \end{pmatrix} \end{array} \quad (12)$$

where S_L denotes the left shift operator. After finding the optimal solution for this problem, one can shift it back to the right to obtain the optimal solution $\phi_E^o(k)$.

Consider now the case where $T_{1E}(z)$ and $T_{2E}(z)$ are proper, but not strictly proper. Clearly, for the $\mathcal{L}_1/\mathcal{H}_2$ problem to have a finite solution, we must have

$$Q_E(\infty) = \frac{T_{1E}(\infty)}{T_{2E}(\infty)}$$

or

$$q_E(0) = \frac{t_{1E}(0)}{t_{2E}(0)}$$

assuming that $t_{2E}(0) \neq 0$. Define:

$$\begin{array}{rcl} \tilde{T}_{1E} & = & T_{1E} - T_{2E}(z)q_E(0) \\ \tilde{T}_{2E} & = & \frac{T_{2E}(z)}{z} \\ \tilde{Q}_E(z) & = & z(Q_E(z) - q_E(0)) \end{array} \quad (13)$$

Then our problem can be written as,

$$\mu_B = \inf_{\tilde{q} \in l_1} ||\tilde{t}_{1B} - \tilde{t}_{2B} * \tilde{q}_B||_{l_1}$$

subject to

$$||\tilde{t}_{1B} - \tilde{t}_{2B} * \tilde{q}_B||_{l_2}^2 \leq \gamma_B{}^2, A_B\phi_B = b_B \quad (14)$$

Note that $\tilde{T}_{1E}(z)$ and $\tilde{T}_{2E}(z)$ are both strictly proper and $\tilde{T}_{1E}(z_i) = T_{1E}(z_i)$ for all unstable zeros of $T_{2E}(z)$. Thus (14) is equivalent to:

$$\mu_B = \inf_{\tilde{q} \in l_1} ||S_L * (\tilde{t}_{1B} - \tilde{t}_{2B} * \tilde{q}_B)||_{l_1}$$

subject to

$$||S_L * (\tilde{t}_{1B} - \tilde{t}_{2B} * \tilde{q}_B)||_{l_2}^2 \leq \gamma_B{}^2, A_B(S_L * \phi_B) = \bar{b}_B \quad (15)$$

Again, the optimal solution $\phi_E^o(k)$ can be recovered by shifting back the optimal solution obtained from the problem (15).

5 An example

Consider the plant $P(s) = \frac{1}{s-1}$. The goal is to design a rational controller $K(s)$ to minimize the $\mathcal{L}_1$ norm of the control action

$$T_{zw}(s) = \frac{K(s)}{1 + P(s)K(s)}$$

subject to the constraint

$$||T_{zw}||_{\mathcal{H}_2}^2 \leq \gamma^2$$

One Youla parameterization is given by

$$\begin{array}{rcl} \Phi(s) & = & T_1(s) - T_2(s)Q(s) \\ & = & \frac{(s-1)(s+5)}{(s+1)(s+2)} - \frac{-(s-1)^2}{(s+1)^2}Q(s) \end{array} \quad (16)$$

For this example the optimal unconstrained $\mathcal{L}_2$ cost is $||\phi^{o,2}||_{\mathcal{L}_2}^2 = 2.828^2$. The optimal unconstrained $\mathcal{L}_1$ cost (found using the method in [6]) is $||\phi^{o,1}||_{\mathcal{L}_1} = 3.6$. Assume that the desired $\mathcal{H}_2$ level is $\gamma^2 = 10$. Choosing $\tau = 0.1^2$ yields

$$\begin{array}{rcl} \Phi_E(z) & = & T_{1E}(z) - T_{2E}(z)Q_E(z) \\ & = & \frac{(z-1.1)(z-0.5)}{(z-0.9)(z-0.8)} - \frac{-(z-1.1)^2}{(z-0.9)^2}Q_E(z) \end{array} \quad (17)$$

Since in this case $T_{1E}(z)$ and $T_{2E}(z)$ are both proper, the modified interpolation constraint must be used. The transformed closed loop map is given by

$$\begin{array}{rcl} \Phi_E(z) & = & \tilde{T}_{1E}(z) - \tilde{T}_{2E}(z)\tilde{Q}_E(z) \\ & = & \frac{0.5(z-1.1)(z-0.9)(z-0.86)}{(z-0.9)^3(z-0.8)} \\ & & -\frac{-(z-1.1)^2}{z(z-0.9)^2}\tilde{Q}_E(z) \end{array} \quad (18)$$

The interpolation conditions are given by $A_E\phi = b_E$ where:

$$\begin{array}{rcl} A_E & = & \begin{pmatrix} 1 & \frac{1}{z_1} & \frac{1}{z_1^2} & \frac{1}{z_1^3} & \cdots \\ 0 & -\frac{1}{z_1^2} & -\frac{2}{z_1^3} & -\frac{3}{z_1^4} & \cdots \end{pmatrix} \\ b_E & = & \begin{pmatrix} T_{1E}(z) \\ \frac{z}{dz}T_{1E}(z) \end{pmatrix} \end{array} \quad (19)$$

[2] stability requires $\tau < 1$

Note that $\frac{z}{dz}\tilde{T}_{1E}(z_i) = \frac{z}{dz}T_{1E}(z_i)$ for all unstable zeros of $\tilde{T}_{2E}(z)$. Let:

$$\begin{aligned}\bar{\Phi}_E(z) &= z\Phi_E(z) = \bar{\tilde{T}}_{1E}(z) - \bar{\tilde{T}}_{2E}(z)\tilde{Q}_E(z) \\ &= \frac{0.5z(z-1.1)(z-0.9)(z-0.86)}{(z-0.9)^3(z-0.8)} \\ &\quad - \frac{-0.5(z-1.1)^2}{(z-0.9)^2}\tilde{Q}_E(z)\end{aligned} \tag{20}$$

After shifting ϕ_E, the interpolation condition can be rewritten as:

$$\begin{aligned}\bar{A}(S_L * \phi_E) &= \bar{b}_E \\ \bar{A}_E &= \begin{pmatrix} 1 & \frac{1}{z_1} & \frac{1}{z_1^2} & \frac{1}{z_1^3} & \cdots \\ -\frac{1}{z_1} & -\frac{2}{z_1^2} & -\frac{3}{z_1^3} & -\frac{4}{z_1^4} & \cdots \end{pmatrix} \\ \bar{b}_E &= \begin{pmatrix} z_1 T_{1E}(z_1) \\ z_1 \frac{z}{dz} T_{1E}(z_1) \end{pmatrix}\end{aligned} \tag{21}$$

In this example $\bar{A}_E$ and $\bar{b}_E$ are given by

$$\bar{A}_E = \begin{pmatrix} 1 & \frac{1}{1.1^1} & \cdots \\ -\frac{1}{1.1^1} & -\frac{2}{1.1^2} & \cdots \end{pmatrix}, \; \bar{b}_E = \begin{pmatrix} 0 \\ 11 \end{pmatrix} \tag{22}$$

Using the method proposed in [10], the problem was reduced to a finite dimensional convex optimization problem. The optimal solution $\bar{\Phi}_E^o(z)$ has order 39 (hence $\Phi_E^o(z)$ has order 40). The corresponding closed-loop transfer function, given by $\Phi(s) = \Phi_E^o(z)|_{z=1+0.1s}$, has $||\phi||^2_{\mathcal{L}_2} = 2.8705$ and $||\phi||_{\mathcal{L}_1} = 5.0339$.

Given the high order of the resulting closed-loop system, the synthesis was followed by a model reduction step, yielding the following 5^{th} controller

$$\frac{4.8s^4 + 65.3s^3 + 448.5s^2 + 1055.3s + 1474.4}{s^5 + 18.3s^4 + 156.0s^3 + 588.2s^2 + 1086.2s + 1016.6}$$

virtually achieving the same performance.

Before close this section we want to briefly address the issue of controller complexity. As illustrated by this example, rational approximations to the optimal $\mathcal{L}_1/\mathcal{H}_2$ controller may have order many times larger than that of the plant (this is not surprising since we are trying to approximate an infinite-dimensional system). While controller reduction methods usually succeed in producing low order controllers, optimality may be lost in the process. As an alternative to the two-tiered method of designing an optimal controller followed by model reduction, suboptimal fixed-order controllers can be synthesized by extending the LMI-based method proposed in [13] to the output feedback case using similar techniques to those in [4]

For this example the LMI-based approach yields the following first order controller

$$K_{LMI}(s) = \frac{4.6225}{s + 3.3}$$

with corresponding $\beta = 3.333$ and $\gamma = 8.1398$. Table 1 compares the performance achieved by the EAS-based controller (after model reduction), the LMI-based controller and the unconstrained optimal $\mathcal{L}_1$ and $\mathcal{H}_2$ controllers. The corresponding impulse responses are shown in Figure 2.

Table 1: Cost for different approaches

Method	$\|\|\phi\|\|_{\mathcal{L}_1}$	$\|\|\phi\|\|_{\mathcal{H}_2}$
EAS($\tau = 0.1$, 5^{th} order)	5.0232	2.8818
LMI (1^{st} order)	5.3086	2.8561
Unconstrained $\mathcal{L}_1$	3.6000	∞
Unconstrained $\mathcal{H}_2$	5.7389	2.8280

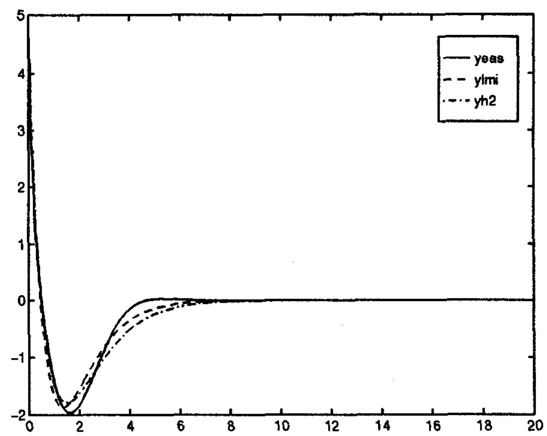

Figure 2: Impulse responses for the two methods

6 Conclusions and directions for further research

In this paper we consider the continuous-time counterpart of the mixed $l_1/\mathcal{H}_2$ problem introduced in [10]. The first part of the paper shows that, contrary to the situation in the discrete-time case where the solution to the problem is well behaved, the continuous-time mixed $\mathcal{L}_1/\mathcal{H}_2$ problem leads to a non-rational transfer function, even when the original plant is rational.

Given the difficulties entailed in physically implementing a non-rational controller, in the second part of the paper we explore the restriction of the problem to rational functions. The main result of this section shows that suboptimal rational controllers can be synthesized by solving an auxiliary discrete-time problem, obtained by considering the Euler Approximating System (EAS) of the plant, together with an additional interpolation constraint, leading to a non-standard $l_1/\mathcal{H}_2$ problem. While these rational controllers can achieve near-optimal performance (as $\tau \rightarrow 0$) they may have high order (many times the order of the plant)[3]. Thus practical considerations often mandate that the synthesis step be followed by model reduction. Alternatively, fixed order suboptimal controllers can be obtained using an LMI-based approach similar to the one that we proposed in [13]. While at this point there are no a-priori bounds on the gap between the performance of these controllers and the optimal, an estimate can be obtained by comparing their performance against that of the EAS-based rational approximations.

Issues still open at present include selecting a-priori the parameter τ in the EAS method in order to obtain a guaranteed level of suboptimality and obtaining explicit expressions for the gap between actual $\mathcal{H}_2$ performance and the upper bound used in the LMI-based method.

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